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The Secretary
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D.V. Postgraduate College Orai 285001, U.P., India
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website : www.vijnanaparishadofinda.org

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ON CERTAIN GAUSS-TYPE QUADRATURE RULES

By

Gradimir V. Milovanović

Serbian Academy of Science and Arts, Begrade & State University of Novi Pazar, Serbia,

E-mail:gmv@mi.sanu.ac.rs

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ABSTRACT

In this short note we give some comments to recent paper by M.A. Bokhari, A. Qadir, and H. Al-Attas, On Gauss-type quadrature rules, Numer. Funct. Anal. Optime. 31 (2010), 1120-1134. Their polynomials are a special case of the Jacobi polynomials on (0,1). In addition we construct orthogonal polynomials $\pi_n(x), n=0,1,\ldots,$ and the corresponding Gaussian quadrature rules with respect to the linear B-spline (as a weight function) and give some numerical examples in order to illustrate an application of such quadratures.

2010 Mathematics Subject Classification : 66D30, 65D32, 33C45

Key Words and Phrases : Orthogonal polynomials, Jacobi polynomials, weight function, three-term recurrence relation, Gaussian quadrature rule, linear B-spline, nodes, Christoffel numbers.

1. Introduction. Recently Bokhari, Qadir, and Al-Attas [1] considered Gauss-type quadrature rules based on polynomials $p_n(t)$ orthogonal on (0,1) with respect to the linear weight function $\omega(t):=1-t$. They discussed a development of $p_n(t)$ via Gaussian hypergeometric differential equation, narrated some of its properties, derived the three-term recurrence relation for the monic polynomials

$$p_{n+1}(t)=\left(t-\frac{2(n+1)^2-1}{4(n+1)^2-1}\right)p_n(t)-\frac{n(n+1)}{4(2n+1)^2}p_{n-1}(t), n=0,1,\ldots \tag{1.1}$$

where $p_0(t)=1$ and $p_{-1}(t)=0$, and considered several numerical examples of such kind of quadratures.

In this short note we show that these polynomials $p_n(t)$ are a special case of the well-known Jacobi polynomials on (0,1). In Section 3 we construct orthogonal polynomials $\pi_n(x), n=0,1,\ldots,$ and the corresponding Gaussian quadrature rules with respect to the linear B-spline (as a weight function). Finally, in Section 4 we

give some numerical examples to illustrate and application of such quadratures.

2. Jacobi Polynomials on (0,1). Let $w(x)=(1-x)^{\alpha}(1+x)^{\beta},\alpha,\beta>-1,$ and $\left\{\hat{P}_n^{(\alpha,\beta)}(x)\right\}$ be a sequence of the corresponding monic Jacobi polynomials, which satisfy the three-term recurrence relation (cf. [8, pp. 131-140])

$$\hat{P}_{n+1}^{(\alpha,\beta)}(x)=(x-\hat{\alpha}_n)\hat{P}_n^{(\alpha,\beta)}(x)-\hat{\beta}_n\hat{P}_{n-1}^{(\alpha,\beta)}(x), n=0,1,\ldots,$$

where $\hat{P}_0^{(\alpha,\beta)}(x)=1,\hat{P}_{-1}^{(\alpha,\beta)}(x)=0$, and

$$\hat{\alpha}_n=\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} \qquad (n\ge 0),$$

$$\hat{\beta}_n=\frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2\left[(2n+\alpha+\beta)^2-1\right]} \qquad (n\ge 1),$$

except when ; then .

By a change of variables $x=2t-1$ we get the monic orthogonal polynomials $p_n^{(\alpha,\beta)}(t)\left(=2^{-n}\hat{P}_n^{(\alpha,\beta)}(2t-1)\right)$ orthogonal on (0,1) with respect to the weight function $\omega(t):=(1-t)^{\alpha}t^{\beta},\alpha,\beta>-1$. The coefficients in their three-term recurrence relation

$$p_{n+1}^{(\alpha,\beta)}(t)=(x-\alpha_n)p_n^{(\alpha,\beta)}(t)-\beta_n p_{n-1}^{(\alpha,\beta)}(t), \quad n=0,1,\ldots, \tag{2.1}$$

are

$$\alpha_n=\frac{1}{2}(1+\hat{\alpha}_n)=\frac{(2n+\alpha+\beta+1)^2-(1+\alpha^2-\beta^2)}{2\left[(2n+\alpha+\beta+1)^2-1\right]}(n\ge 0),$$

$$\beta_n=\frac{1}{4}\hat{\beta}_n=\frac{n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2\left[(2n+\alpha+\beta)^2-1\right]} \quad (n\ge 1).$$

For $\alpha=1$ and $\beta=0$, this relation (2.1) gives orthogonal polynomials $p_n(t)$ discussed in [1], which satisfy the relation (1.1).

Now, we list the parameters $\alpha_n(n\ge 0)$ and $\beta_n(n\ge 1)$ for some other special cases:

(a) $\alpha=0,\beta=1$.

$$\alpha_n=\frac{2(n+1)^2}{(2n+1)(2n+3)},\beta_n=\frac{n(n+1)}{4(2n+1)^2};$$

(b) $\alpha = 1, \beta = 1.$

$$\alpha_n = \frac{1}{2}, \beta_n = \frac{n(n+2)}{4(2n+1)(2n+3)};$$

(c) $\alpha = 2, \beta = 0.$

$$\alpha_n = \frac{n^2+3n+1}{2(n+1)(n+2)}, \beta_n = \frac{n^2(n+2)^2}{4(n+1)^2(2n+1)(2n+3)};$$

(d) $\alpha = 2, \beta = 1.$

$$\alpha_n = \frac{2(n+1)(n+3)}{(2n+3)(2n+5)}, \beta_n = \frac{n(n+3)}{4(2n+3)^2};$$

(e) $\alpha = 2, \beta = 2.$

$$\alpha_n = \frac{1}{2}, \beta_n = \frac{n(n+4)}{4(2n+3)(2n+5)};$$

(f) $\alpha = 3, \beta = 0.$

$$\alpha_n = \frac{2n^2+8n+3}{(2n+3)(2n+5)}, \beta_n = \frac{n^2(n+3)^2}{4(n+1)(n+2)(2n+3)^2};$$

(g) $\alpha = 3, \beta = 1.$

$$\alpha_n = \frac{(n+1)(n+4)}{2(n+2)(n+3)}, \beta_n = \frac{n(n+1)(n+3)(n+4)}{4(n+2)^2(2n+3)(2n+5)};$$

(h) $\alpha = \beta = -1/2.$

$$\alpha_n = \frac{1}{2}, \beta_1 = \frac{1}{8}, \beta_n = \frac{1}{16}(n \geq 2), \text{ etc.}$$

Remark 2.1. The relation (2.1) can be obtained taking $w(x) = |x|^\gamma (1-x^2)^\alpha$, with and the corresponding generalized Gegenbauer polynomials , where , which were introduced by Lasčěnov [7] (see, also, [2, pp. 155-156] and [8, pp. 147-148]). Their three-term recurrence relation is

$$W_{n+1}^{(\alpha,\beta)}(x) = xW_n^{(\alpha,\beta)}(x) - B_n W_{n-1}^{(\alpha,\beta)}(x), W_0^{(\alpha,\beta)}(x) = 1, W_{-1}^{(\alpha,\beta)}(x) = 0,$$

with recursion coefficients

$$B_{2n} = \frac{n(n+\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, B_{2n-1} = \frac{(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)},$$

except $\alpha+\beta=-1$; then $B_1 = \beta+1$

Since the weight function is even on (–1,1), using Theorems 2.2.11 and 2.2.12 from [8, pp. 102-103], we get (2.1) for polynomials orthogonal with respect to the weight $\omega(t)=w(\sqrt{t})/\sqrt{t}=t^{\beta}(1-t)^{\alpha}$, with $\alpha_0=B_1=(\beta+1)/(\alpha+\beta+2)$, $\alpha_n=B_{2n}+B_{2n+1}$, $\beta_n=B_{2n-1}B_{2n}, n\geq 1$.

3. A Gaussian quadrature formula. Sometimes in applications it could be of some interest to construct orthogonal polynomials $\pi_n(x), n=0,1,\ldots,$ and the corresponding Gaussian quadrature rules with respect to the linear B-spline (as a weight function)

$$w(x)=B_1(x)=\begin{cases}1+x, & -1\leq x\leq 0\\ 1-x & 0<x\leq 1\\ 0, & \text{otherwise,}\end{cases}$$

i.e.,

$$\int_R f(x)B_1(x)dx=\int_{-1}^{1} f(x)(1+|x|)dx=\sum_{k=1}^{n}A_k f(x_k)+R_n(f), \tag{3.1}$$

where $R_n(p)=0$ for all polynomials p of degree at most $2n-1$.

This weight function is an even extension of $\omega(t)=1-t$ from (0,1) to (-1,1). The coefficients in the three-term recurrence relation for the corresponding monic orthogonal polynomials $\pi_n(x)$,

$$\pi_{n+1}(x)=x\pi_n(x)-\beta_n\pi_{n-1}(x), n=1,2,\ldots$$

are

$\beta_0=1, \beta_1=1/6, \beta_2=7/30, \beta_3=57/245, \beta_4=683/2793, \beta_5=207725/856482,$

$\beta_6=286749501/1159331030, \beta_7=286268318986/1164429355245,$

$\beta_8=272609711230510/1097298927604497$, etc.

For example,

$$\pi_0(x)=1, \pi_1(x)=x, \pi_2(x)=x^2-\frac{1}{6}, \pi_3(x)=x^3-\frac{2x}{5}, \pi_4(x)=x^4-\frac{31x^2}{49}+\frac{19}{490},$$

$$\pi_5(x)=x^5-\frac{50x^3}{57}+\frac{109x}{798}, \pi_6(x)=x^6-\frac{16825x^4}{15026}+\frac{2179x^2}{7513}-\frac{5935}{631092}, \text{ etc.}$$

In numerical construction we use our Mathematica Package "Orthogonal Polynomials" [3]. In order to construct quadrature rules up to n points we need the moment $\mu_k = \int_R x^k B_1(x)\,dx, k = 0,1,\ldots,2n-1$, which are in our case given by

$$\mu_k = \frac{1+(-1)^k}{(k+1)(k+2)}, k \ge 0.$$

By the mentioned Package, in this case, we can obtain the recursion coefficients, α_k (=0 in this symmetric case) and , in a symbolic form for a reasonable n:
{alpha, beta}=aChebyshevAlgorithm[moments, Algorithm->Symbolic];
taking, for example, the first 800 moments,

moments= Table [(1+(-1)^k)/((1+k)(2+k)), {k,0,800}];

This enables us to construct parameters in Gaussian quadrature

$$Q_n(f) = \sum_{k=1}^{n} A_k^{(n)} f\left(x_k^{(n)}\right)$$

up to $n \le 400$ nodes. For example, for n=100, with Precision ->40, the statement is the following

```
{n100,w100}=aGaussianNodesWeigths[100,alpha,beta,
                    WorkingPrecision->50, Precision->40];
```

where n100 and w100 are sequences of nodes $x_k^{(n)}, k = 1,\ldots,n$, and Christoffel numbers $A_k^{(n)}, k = 1,\ldots,n$, respectively. The last command implements the wellknown Golub-Welsch algorithm [6].

The corresponding software in Matlab was given by Gautschi [4], The first FORTRAN package of routines ORTHPOL was also developed by Walter Gautschi [5] in 1994.

4. Numerical results. In this section we consider three integrals

$$I_k = \int_R f_k(x) B_1(x)\,dx, k = 1,2,3,$$

for the functions $f_1(x) = \cos(\pi x)$,

$$f_2(x) = \frac{1}{\left(x+\frac{3}{10}\right)^2 + \frac{1}{100}} + \frac{1}{\left(x-\frac{9}{10}\right)^2 + \frac{1}{25}} \text{ and } f_3(x) = \frac{2}{2+\sin(10\pi x)},$$

and then apply the quadrature formula (3.1) for their calculation. The first function is smooth, the second is quasi-singular, and the last is an oscillatory function. The graphics of integrands $f_k(x)B_1(x), k = 1,2,$ are displayed in Fig. 4.1, and the

graphic of the oscillatory function $f_3(x)B_1(x)$ is presented in Fig. 4.2

Using our Mathematica Package "Orthogonal Polynomials" [5] we constructed n-point Gaussian rules for $n=5$; 10(10)50; 100(50)400. Then, we applied these rules to integrals $I_k, k=1,2,3$, and compared the obtained results with the exact values

$$I_1 = 4/\pi^2 = 0.40528473456935108577551785283891055561 7435...$$

$$I_2 = 22.3859074908351131640822746542985712503464...$$

$$I_3 = 1.1547005383792515290182975610039149112952 0...$$

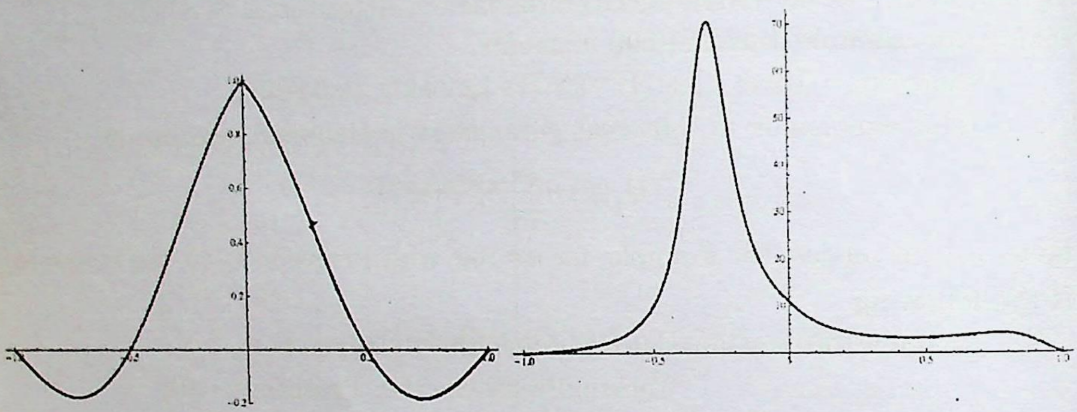


Figure 4.1. *Graphics of function* $x \to B_1(x) f_1(x)$ *(left) and* $x \to B_1(x) f_2(x)$ *(right)*

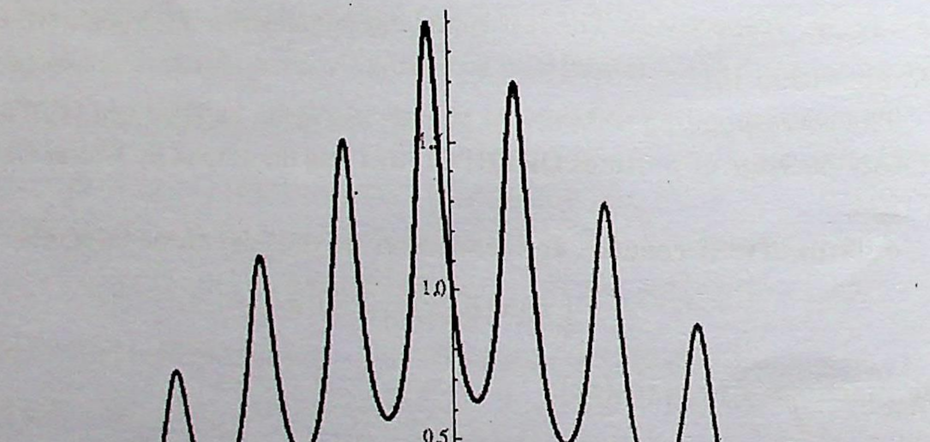


Figure 4.2. Graphics of the function $x \to B_1(x) f_3(x)$

In Table 4.1 we give the relative error in Gaussian approximations

$$REL_k(n)=\left|\frac{Q_n(f_k)-I_k}{I_k}\right|, k=1,2,3.$$

Numbers in parentheses indicate decimal exponents.

Table 4.1. Relative errors $REL_k(n)$ in Gaussian approximations $Q_n(f_k)$ for $k=1,2,3$

n	$REL_1(n)$	$REL_2(n)$	$REL_3(n)$
5	2.89(–5)	3.33(–1)	1.39(–2)
10	4.01(–15)	1.25(–1)	2.97(–2)
20	1.01(–40)	1.81(–2)	7.79(–2)
30		2.50(–3)	1.39(–2)
40		3.26(–4)	3.69(–4)
50		4.05(–5)	3.32(–4)
100		4.22(–10)	2.68(–6)
150		2.28(–14)	1.05(–6)
200		9.20(–19)	3.63(–9)
250		4.38(–24)	3.79(–10)
300		6.49(–28)	1.49(–12)
350		1.99(–32)	1.81(–14)
400		3.40(–38)	7.64(–16)

As we can see in the case of a smooth function f_1 the convergence is very fast. The relative error is about 10^{-40} for $n=20$ points. As we expected the convergence in the last case is very slow. We must use 400 points in the quadrature rule in order to obtain the so-called double precision result (with about 16 decimal digits).

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HARDY AND COWLING-PRICE THEOREMS FOR A CHEREDNIK TYPE OPERATOR ON THE REAL LINE

By

Mohammad Ali Mourou

Department of Mathematics, College of Sciences for Girls University of Dammam, P.O. 1982, Damman 31441, Kingdom of Saudi Arabia

E-mail : mohamed_ali.mourou@yahoo.fr

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ABSTRACT

This paper is aimed to establish Hardy and Cowling-Price type theorems for the Fourier transform tied to a generalized Cheredink operator on the real line.

2010 Mathematics Subject Classificàtion : Primary 33C45, 43A15, 43A32; Secondary 44A15.

Key Words and Phrases : Differential-difference operator; Generalized Fourier transform; Hardy and Cowling-Price theorems.

1. Introduction. In his 1933 paper [8]. Hardy obtained the following famous theorem :

Theorem 1.1. Let $1 \le p,q \le \infty$ with at least one of them finite. Let f be a measurable function on $\mathbb{R}$ such that

$$e^{ax^2} f \in L^p(\mathbb{R}) \text{ and } e^{bx^2}\mathcal{F}_u(f) \in L^q(\mathbb{R}), \tag{1}$$

for some positive constants a and b. Then

- If $ab \ge 1/4$, we have $f=0$ almost everywhere
- If $ab < 1/4$, there are infinitely many nonzero functions satisfying (1).

Above $\mathcal{F}_u$ stands for the ordinary Fourier transform on $\mathbb{R}$ given by

$$\mathcal{F}_u(f)(\lambda) = \int_{\mathbb{R}} f(x) e^{-i\lambda x} dx.$$

Later, Cowling and Price [4] obtained the following L^p version of Theorem 1.1:

Theorem 1.2. Let f be a measurable function on $\mathbb{R}$ such that

$$e^{ax^2} f \in L^\infty(\mathbb{R}) \text{ and } e^{b\lambda^2}\mathcal{F}_u(f) \in L^\infty(\mathbb{R}), \tag{2}$$

for some positive constants a and b. Then

- If $ab > 1/4$, we have $f=0$ almost everywhere.
- If $ab = 1/4$, the function f is of the form $f(x) = c_0 e^{-ax^2}, c_0 \in \mathbb{C}$.

- If $ab<1/4$, there are infinitely many nonzero functions satisfying (2).

Many generalizations of Theorems 1.1 and 1.2 to new contexts have been discovered. For instance, these theorems have been obtained in [2] for semi-simple Lie groups, in [5] for the motion group and in [15] for Chébli-Triméche hypergroups.

The intention of this paper is to establish analogues of Theorems 1.1 and 1.2 when in (1) and (2) the usual Fourier transform $\mathcal{F}_u$ is substituted by a generalized Fourier transform $\mathcal{F}_\Lambda$ on $\mathbb{R}$ associated with a the first-order singular differential-difference operator:

$$\Lambda f(x)=\frac{df}{dx}+\frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right)-\rho f(-x),$$

where

$$A(x)=|x|^{2\alpha+1}B(x),\ \ \alpha>-1/2,$$

B being a positive C^∞ even function on $\mathbb{R}$, and $\rho=0$. In addition we suppose that

(i) A is increasing on $[0,\infty[$ and $\lim_{x\to\infty}A(x)=\infty$;

(ii) A'/A is decreasing on $]0,\infty[$ and $\lim_{x\to\infty}A'(x)/A(x)=2\rho$;

(iii) There exists a constant $\delta>0$ such that the function $e^{\delta_x}\left(A'(x)/A(x)-2\rho\right)$ is bounded for large $x>0$ together with its derivatives.

Notice that the differential-difference operator

$$D_{\alpha,\beta}f(x)=\frac{df}{dx}+\left[(2\alpha+1)\coth x+(2\beta+1)\tanh x\right]\left(\frac{f(x)-f(-x)}{2}\right)-(\alpha+\beta+1)f(-x),$$

which is referred to as the Jacobi-Cherednik operator (see [7] is of the same type as Λ with

$$\begin{cases}A(x)=(\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1};\alpha\geq\beta>-1/2;\\ \rho=\alpha+\beta+1;\ \ \delta=2.\end{cases}$$

The one-dimensional Cherednik operator (see [3]) is a particular case of $D_{\alpha,\beta}$. Such operators have been used by Heckmann and Opdam to develop a theory generalizing harmonic analysis on symmetric spaces (see [9,12].) For recent important results in this direction we refer to [13,16,17].

The author [11] has initiated a quite new commutative harmonic analysis on the real line related to the differential-difference operator Λ in which several analytic structures on $\mathbb{R}$ were generalized. The tools actually required for the

discussion in the present paper, are essentially the Fourier transform and the Gaussian kernel tied to Λ.

2. Preliminaries. In [11] we have shown that for each $\lambda \in \mathbb{C}$, the differential-difference equation

$$\Lambda u = i\lambda u, \qquad u(0) = 1,$$

admits a unique C^∞ solution on $\mathbb{R}$, denoted Φ_λ given by

$$\Phi_\lambda(x) = \begin{cases} \varphi_\lambda(x) + \dfrac{1}{i\lambda - \rho}\dfrac{d}{dx}\varphi_\lambda(x) & \text{if } \lambda \neq -i\rho, \\ 1 + \dfrac{2\rho}{A(x)}\displaystyle\int_0^x A(t)\,dt & \text{if } \lambda = -i\rho, \end{cases} \tag{3}$$

where φ_λ denotes the solution of the differential equation

$$\Delta u = -(\lambda^2 + \rho^2)u, \qquad u(0) = 1, \; u'(0) = 1, \tag{4}$$

Δ being the second-order singular differential operator defined by

$$\Delta = \frac{1}{A(x)}\frac{d}{dx}\left(A(x)\frac{d}{dx}\right). \tag{5}$$

Moreover, $\Phi_\lambda(x)$ is entire in λ.

Remark 2.1. For $A(x) = (\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}, \alpha \geq \beta > -1/2$, the differential operator Δ reduces to the so-called Jacobi operator. The eigenfunction φ_λ is given by

$$\varphi_\lambda(x) = {}_2F_1\left(\frac{\alpha+\beta+1+i\lambda}{2}, \frac{\alpha+\beta+1-i\lambda}{2}; \alpha+1; -(\sinh x)^2\right)$$

where ${}_2F_1$ is the Gauss hypergeometric function [10].

From [14] recall the following estimates.

Lemma 2.1. (i) For every $x \in \mathbb{R}$,

$$e^{-\rho|x|} \leq \varphi_0(x) \leq 1. \tag{6}$$

(ii) There is a constant $C > 0$ such that

$$\left|\frac{d^n}{d\lambda^n}\varphi_\lambda(x)\right| \leq C(1+|x|)|x|^n e^{(|\mathrm{Im}\,\lambda|-\rho)|x|}, \tag{7}$$

for all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and $n=0,1,\ldots$

Analogous estimates for $\Phi_\lambda(x)$ are provided by the next statement.

Proposition 2.1. There is a constant $C>0$ such that

$$\left|\frac{d^n}{d\lambda^n}\Phi_\lambda(x)\right| \le C(1+|\lambda|)(1+|x|)^2 |x|^n e^{(|\mathrm{Im}\,\lambda|-\rho)|x|}, \tag{8}$$

for all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and $n=0,1\ldots$

Proof. By (3),

$$\frac{d^n}{d\lambda^n}\Phi_\lambda(x) = \frac{d^n}{d\lambda^n}\varphi_\lambda(x) + \frac{d^n}{d\lambda^n}\left(\frac{1}{i\lambda-\rho}\frac{d}{dx}\varphi_\lambda(x)\right).$$

As by (4),

$$\frac{d}{dx}\varphi_\lambda(x) = -\mathrm{sgn}(x)\frac{\lambda^2+\rho^2}{A(x)}\int_0^{|x|}\varphi_\lambda(t)A(t)\,dt, \tag{9}$$

we obtain

$$\frac{d^n}{d\lambda^n}\left(\frac{1}{i\lambda-\rho}\frac{d}{dx}\varphi_\lambda(x)\right) = \frac{\mathrm{sgn}(x)}{A(x)}\int_0^{|x|}\frac{d^n}{d\lambda^n}\left[(i\lambda+\rho)\varphi_\lambda(t)\right]A(t)\,dt.$$

The result follows now from (7) and Leibniz formula.

Note 2.1. For a function f on $\mathbb{R}$, write $f_e(x) = (f(x)+f(-x))/2$ and $f_0(x) = (f(x)-f(-x))/2$ respectively for its even and old parts. We denote by

- $\mathcal{S}(\mathbb{R})$ the space of C^∞ function f on $\mathbb{R}$ which are rapidly decreasing together with their derivatives, i.e., such that for all $m,n=0,1,\ldots,$

$$P_{m,n}(f) = \sup_{x\in\mathbb{R}}(1+x^2)^m\left|\frac{d^n}{dx^n}f(x)\right| < \infty.$$

The topology of $\mathcal{S}(\mathbb{R})$ is defined by the semi-norms $P_{m,n}, m,n=0,1,\ldots$

- $\mathcal{S}_e(\mathbb{R})$ (resp. $\mathcal{S}_0(\mathbb{R})$) the subspace of $\mathcal{S}(\mathbb{R})$ consisting of even (resp. odd) functions.
- $\mathcal{S}^2(\mathbb{R})$ the space of C^∞ functions f on $\mathbb{R}$ such that for all $m,n=0,1,\ldots$

$$Q_{m,n}(f) = \sup_{x\in\mathbb{R}}(1+x^2)^m\left|\frac{d^n}{dx^n}f(x)\right| < \infty.$$

The topology of $\mathcal{S}^2(\mathbb{R})$ is defined by the semi-norms $Q_{m,n}, m,n=0,1,\ldots$

- $\mathcal{S}_e^2(\mathbb{R})$ (resp. $\mathcal{S}_0^2(\mathbb{R})$) the subspace of $\mathcal{S}^2(\mathbb{R})$ consisting of even (resp. odd) functions.
- $\mathcal{I}$ the map defined by $\mathcal{I}h(x)=\int_{-\infty}^{x} h(t)dt, x\in\mathbb{R}$.

Remark 2.2. (i) By (6) we see that $\mathcal{S}^2(\mathbb{R})\subset\mathcal{S}(\mathbb{R})$.

(ii) It is easily checked that $\mathcal{S}^2(\mathbb{R})$ is invariant under the differential-difference operator Λ.

(iii) Due to our assumptions on the function A there is a positive constant k such that

$$A(x)\sim ke^{2\rho|x|} \text{ as } |x|\to\infty. \tag{10}$$

The following technical lemma will be useful.

Lemma 2.2. The map $\mathcal{I}$ is a topological isomorphism from $\mathcal{S}_o^2(\mathbb{R})$ onto $\mathcal{S}_e^2(\mathbb{R})$.

Proof. It is sufficient to show that $\mathcal{I}$ maps continously $\mathcal{S}_0^2(\mathbb{R})$ into $\mathcal{S}_e^2(\mathbb{R})$. Let $f\in S_0^2(\mathbb{R})$. Clearly $\mathcal{I}f$ is a C^∞ even function on $\mathbb{R}$. For $n=1,2,\ldots$, $Q_{m,n}(\mathcal{I}f)=Q_{m,n-1}(f)$. Moreover, as by (9), φ_0 is decreasing on $[0,\infty[$, we get

$$\begin{aligned}\left(1+x^2\right)^m\varphi_0(x)^{-1}|\mathcal{I}f(x)| &\le \left(1+x^2\right)^m\varphi_0(x)^{-1}\int_{|x|}^{\infty}|f(t)|dt\\ &\le \int_{|x|}^{\infty}\left(1+t^2\right)^m\varphi_0(t)^{-1}|f(t)|dt\\ &\le Q_{m+1,0}(f)\int_{|x|}^{\infty}\frac{dt}{\left(1+t^2\right)}.\end{aligned}$$

Hence $Q_{m,0}(\mathcal{I}f)\le\frac{\pi}{2}Q_{m+1,0}(f)$. This ends the proof.

The generalized Fourier transform of a suitable function f on $\mathbb{R}$ is defined by

$$\mathcal{F}_\Lambda(f)(\lambda)=\int_{\mathbb{R}} f(x)\Phi_{-\lambda}(x)A(x)dx,\ \lambda\in\mathbb{R}.$$

Remark 2.3. According to (7), (8) and (10), the generalized Fourier transform $\mathcal{F}_\Lambda$ is well defined on $S^2(\mathbb{R})$.

Proposition 2.2. For all $f\in S^2(\mathbb{R})$,

$$\mathcal{F}_\Lambda(f)(\lambda)=\mathcal{F}_\Delta(f_e)(\lambda)+(i\lambda-\rho)\mathcal{F}_\Delta J(f_0)(\lambda), \tag{11}$$

where $\mathcal{F}_\Delta$ stands for the Fourier transform related to the differential operator Δ, defined on $S_e^2(\mathbb{R})$ by

$$\mathcal{F}_\Delta(h)(\lambda)=\int_{\mathbb{R}} h(x)\varphi_\lambda(x)A(x)dx,\ \lambda\in\mathbb{R}.$$

Proof. If $f\in S_e^2(\mathbb{R})$, identity (11) is obvious. Assume $f\in S_0^2(\mathbb{R})$. By using (3),(4),(5) and by integrating by parts we obtain

$$\mathcal{F}_\Delta(f)(\lambda)=\frac{-1}{i\lambda+\rho}\int_{\mathbb{R}} f(x)\varphi_\lambda'(x)A(x)dx$$

$$=\frac{1}{i\lambda+\rho}\int_{\mathbb{R}} \mathcal{I}f(x)\left(A(x)\varphi_\lambda'(x)\right)'dx$$

$$=\frac{1}{i\lambda+\rho}\int_{\mathbb{R}} \mathcal{I}f(x)\Delta\varphi_\lambda(x)A(x)dx$$

$$=(i\lambda-\rho)\int_{\mathbb{R}} \mathcal{I}f(x)\varphi_\lambda(x)A(x)dx$$

$$=(i\lambda-\rho)\mathcal{F}_\Delta(Jf)(\lambda),$$

which completes the proof.

Remark 2.4. For $A(x)=(\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}, \alpha\geq\beta>-1/2$, the transform $\mathcal{F}_\Delta$ coincides with the Jacobi transform of order (α,β) (see [10]).

Theorem 2.1. The generalized Fourier transform $\mathcal{F}_\Lambda$ is a topological isomorphism between $S^2(\mathbb{R})$ and $S(\mathbb{R})$. Moreover,

$$\mathcal{F}_\Lambda^{-1}(g)(x)=\mathcal{F}_\Delta^{-1}(g_e)(x)+\left(\rho I+\frac{d}{dx}\right)\mathcal{F}_\Delta^{-1}\left(\frac{g_o}{i\lambda}\right)(x)$$

of all $g\in S(\mathbb{R})$.

Proof. By [14] we know that the tranform $\mathcal{F}_\Delta$ is a topological isomorphism from $S_e^2(\mathbb{R})$ onto $S_e(\mathbb{R})$. Then the result follows from (11), Lemma 2.2 and the fact that the map $f\to\lambda f$ is a topological isomorphism from $S_e(\mathbb{R})$ onto $S_o(\mathbb{R})$. The identity

above follows easily from (11).

Note 2.2. We denote by

- $\mathcal{D}_a(\mathbb{R}), a>0,$ the space of C^∞ functions on $\mathbb{R}$ supported in $[-a,a]$, provided with the topology of compact convergence for all derivatives.

- $\mathcal{D}(\mathbb{R}) = \bigcup_{a>0} \mathcal{D}(\mathbb{R})$ endowed with the inductive limit topology.

- $H_{a,} a>0,$ the space of entire, rapidly decreasing functions of exponential type a; that is, $f \in \mathcal{H}_a$ if and only if, f is entire on $\mathbb{C}$ and for all $m = 0, 1, \ldots..,$

$$p_m(f) = \sup_{\lambda \in \mathbb{C}} \left| (1+\lambda)^m f(\lambda) e^{-a|\mathrm{Im}\,\lambda|} \right| < \infty .$$

H_a is equipped with the topology defined by the semi-norms $p_{m,} m = 0,1,\ldots$.

- $H = \bigcup_{a>0} H_a$, equipped with the inductive limit topology.

Another standard result for the generalized Fourier transform $\mathcal{F}_\Lambda$ is as follows.

Theorem 2.2. (Paley-Wiener) The generalized Fourier transform $\mathcal{F}_\Lambda$ is a topological isomorphism from $\mathcal{D}(\mathbb{R})$ onto H. More precisely, $f \in \mathcal{D}_a(\mathbb{R})$ if, and only if, $\mathcal{F}_\Lambda(f) \in H_a$.

According to [11] the inverse generalized Fourier transform $\mathcal{F}_\Lambda^{-1}$ may also be expressed as follows.

Theorem 2.3. For all $g \in \mathcal{S}(\mathbb{R})$,

$$\mathcal{F}_\Lambda^{-1}(g)(x) = \int_{\mathbb{R}} g(\lambda) \Phi_{-\lambda}(-x) d\sigma(\lambda),$$

with

$$d\sigma(\lambda) = \left(\frac{\lambda - i\rho}{\lambda} \right) \frac{d\lambda}{2\pi |c(|\lambda|)|^2}, \tag{12}$$

where $c(s)$ is a continuous function on $]0,\infty[$ such that

$$c(s)^{-1} \sim k_1 s^{\alpha+1/2} \qquad \text{as } s \to \infty, \tag{13}$$

$$c(s)^{-1} \sim k_2 s, \qquad \text{as } s \to 0,$$

for some $k_1, k_2 \in \mathbb{C}$.

Remark 2.5. (i) The tempered measure σ is called the spectral measure associated with the differential difference operator Λ.

(ii) For $A(x)=(\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}, \alpha\geq\beta>-1/2$, we have

$$c(s)=\frac{2^{\alpha+\beta+2-is}\Gamma(is)\Gamma(\alpha+1)}{\Gamma[(\alpha+\beta+1+is)/2]\Gamma[(\alpha-\beta+1+is)/2]},\quad s>0.$$

The next statement provides a Plancherel type formula for the generalized Fourier trasform $\mathcal{F}_\Lambda$.

Theorem 2.4. For all $f,g\in S^2(\mathbb{R})$,

$$\int_{\mathbb{R}} f(x)g(-x)A(x)dx=\int_{\mathbb{R}}\mathcal{F}_\Lambda(f)(\lambda)\mathcal{F}_\Lambda(g)(\lambda)d\sigma(\lambda).$$

Proof. By (11) and (12),

$$\int_{\mathbb{R}}\mathcal{F}_\Lambda(f)(\lambda)\mathcal{F}_\Lambda(g)(\lambda)d\sigma(\lambda)=\int_{\mathbb{R}}\mathcal{F}_\Delta(f_e)(\lambda)\mathcal{F}_\Delta(g_e)(\lambda)\frac{d\lambda}{|c(|\lambda|)|^2}$$

$$-\int_{\mathbb{R}}(\lambda+\rho^2)\mathcal{F}_\Delta I(f_0)(\lambda)\mathcal{F}_\Delta I(g_0)(\lambda)\frac{d\lambda}{|c(|\lambda|)|^2}$$

$$=\kappa_1+\kappa_2.$$

By a Plancherel formula for the transform $\mathcal{F}_\Delta$ (see [14]).

$$\kappa_1=\int_{\mathbb{R}} f_e(x)g_e(x)A(x)dx$$

Moreover, by using the identity

$$\mathcal{F}_\Delta(\Delta h)(\lambda)=-(\lambda^2+\rho^2)\mathcal{F}_\Delta(h)(\lambda),\qquad h\in S_e^2(\mathbb{R}),$$

(see [14]), and by integrating by parts we get

$$\kappa_2=\int_{\mathbb{R}}\mathcal{F}_\Delta(\Delta\mathcal{I}f_0)(\lambda)\mathcal{F}_\Delta(\mathcal{I}g_0)(\lambda)\frac{d\lambda}{|c(|\lambda|)|^2}$$

$$=\int_{\mathbb{R}}\Delta\mathcal{I}(f_0)(x)\mathcal{I}(g_0)(x)A(x)dx$$

$$=\int_{\mathbb{R}}(Af_0)'(x)\mathcal{I}(g_0)(x)dx$$

$$=-\int_{\mathbb{R}} f_0(x)g_0(x)A(x)dx.$$

Hence

$$\kappa_1+\kappa_2=\int_{\mathbb{R}}\left[f_e(x),g_e(x)-f_0(x)g_0(x)\right]A(x)dx\int_{\mathbb{R}}f(x)g(-x)A(x)dx.$$

This completes the proof.

Note 2.3. We denote by

- $L^p(\mathbb{R},A(x)dx), 1\le p\le\infty$, the class of measurable functions f on $\mathbb{R}$ for which $\|f\|_{p,A}<\infty$, where

$$\|f\|_{p,|\sigma|}=\left(\int_{\mathbb{R}}|f(x)|^p A(x)dx\right)^{1/p}, \qquad \text{if } p<\infty,$$

and $\|f\|_{\infty,|\sigma|}=\|f\|_\infty$.

* $L^p(\mathbb{R},|\sigma|), 1\le p\le\infty$, be the class of measurable functions f on $\mathbb{R}$ for which $\|f\|_{p,|\sigma|}<\infty$, where

$$\|f\|_{p,|\sigma|}=\left(\int_{\mathbb{R}}|f(\lambda)|^p d|\sigma|(\lambda)\right)^{1/p}, \qquad \text{if } p<\infty,$$

and $\|f\|_{\infty,|\sigma|}=\|f\|_\infty$.

Remark 2.6. By (8) there is a positive constant $k>0$ such that

$$\left|\mathcal{F}_\Lambda(f)(\lambda)\right|\le k(1+|\lambda|)\|f\|_{1,A}$$

for all $f\in L^1(\mathbb{R},A(x)dx)$.

Lemma 2.3. For all $f\in L^1(\mathbb{R},A(x)dx)$ and $g\in\mathcal{D}(\mathbb{R})$.

$$\int_{\mathbb{R}}f(x)g(-x)A(x)dx=\int_{\mathbb{R}}\mathcal{F}_\Lambda(f)(\lambda)\mathcal{F}_\Lambda(g)(\lambda)d\sigma(\lambda).$$

Proof. Fix $g\in\mathcal{D}(\mathbb{R})$. For $f\in L^1(\mathbb{R},A(x)dx)$ put

$$l_1(f)=\int_{\mathbb{R}}f(x)g(-x)A(x)dx$$

and

$$l_2(f)=\int_{\mathbb{R}}\mathcal{F}_\Lambda(f)(\lambda)\mathcal{F}_\Lambda(g)(\lambda)d\sigma(\lambda).$$

In view of Theorem 2.4; $l_1(f)=l_2(f)$ for each $f\in\mathcal{D}(\mathbb{R})$. Moreover,

$$\left|l_1(f)\right| \leq \|g\|_\infty \|g\|_{1,A}$$

and

$$\left|l_2(f)\right| \leq k\|f\|_{1,A} \int_{\mathbb{R}} \left|\mathcal{F}_\Lambda(g)(\lambda)\right|(1+|\lambda|)\, d|\sigma|(\lambda)$$

by virtue of Remark 2.6. This shows that the linear functionals l_1 and l_2 are bounded on $L^1(\mathbb{R}, A(x)dx)$. Therefore $l_1 = l_2$, and the lemma is proved.

An immediate consequence of the lemma above is

Corollary 2.1. The generalized Fourier transform $\mathcal{F}_\Lambda$ is injective on $L^1(\mathbb{R}, A(x)dx)$.

For $t>0$, the Gaussian kernel E_t associated with the differential-difference operator is Λ defined by

$$E_t(x) = \int_{\mathbb{R}} e^{-t\left(\lambda^2+\rho^2\right)} \phi_{-\lambda}(-x)\, d\sigma(\lambda), \quad x \in \mathbb{R} \tag{14}$$

This kernel enjoys the following properties.

Proposition 2.3. (i) $E_t \in \mathcal{S}^2(\mathbb{R})$ and

$$\mathcal{F}_\Lambda(E_t)(\lambda) = e^{-t\left(\lambda^2+\rho^2\right)}, \text{ for all } \lambda \in \mathbb{R}. \tag{15}$$

(ii) E_t is even, positive and $\int_{\mathbb{R}} E_t(x) A(x)\, dx = 1$.

(iii) The function $u(x,t) = E_t(x)$ is C^∞ on $\mathbb{R} \times]0,\infty[$ and solves the partial differential equation

$$\Delta_x u(x,t) = \frac{\partial}{\partial t} u(x,t),$$

where Δ is given by (5).

(iv) There are two positive constants $C_1(t)$ and $C_2(t)$ such that

$$C_1(t)\frac{e^{-\frac{x^2}{rt}}}{\sqrt{B(x)}} \leq E_t(x) \leq C_2(t)\frac{e^{-\frac{x^2}{4t}}}{\sqrt{B(x)}} \tag{16}$$

(v) Let $p \in [0,\infty[$. Then there exists a positive constant $M(p,t)$ such that

$$\left(E_t(x)\right)^p \le M(p,t)E_{t/p}(x), \tag{17}$$

Proof. Assertion (i) follows directly from Theorems 2.1 and 2.3. A combination of (3), (12) and (14) yields

$$E_t(x) = \int_0^\infty e^{-t\left(\lambda^2+\rho^2\right)} \varphi_\lambda(x) \frac{d\lambda}{\pi\left|c(\lambda)\right|^2} \tag{18}$$

But according to [6], the right hand side of (18) satisfies (ii), (iii) and (iv). According to our assumptions on the function A, there is a constant $k>0$ such that $B(x) \ge k$ for all $x \in \mathbb{R}$. The majorization (17) is then an easy consequence of (16).

3. Hardy and Cowling-Price theorems. The following technical lemmas will greatly simplify the proofs of our main theorems.

Lemma 3.1. [1] Let g be an entire function on $\mathbb{C}$. Suppose that

$$\left|g(x)\right| \le M\left(1+|z|\right)^m e^{a(\operatorname{Re} z)^2} \text{ for all } z \in \mathbb{C}$$

and

$$\left|g(x)\right| \le M \text{ for all } x \in \mathbb{R},$$

for some a, $M>0$ and $m \in \mathbb{N}$. Then g is constant on $\mathbb{C}$.

Lemma 3.2. [1] Let $q \in [1,\infty[$ and g be an entire function on $\mathbb{C}$. Suppose that

$$\int_{\mathbb{R}} \left|g(x)\right|^q dx < \infty$$

and

$$\left|g(x)\right| \le M\left(1+|z|\right)^m e^{a(\operatorname{Re} z)^2} \text{ for all } z \in \mathbb{C},$$

for some $a,M>0$ and $m \in \mathbb{N}$. Then $g=0$ on $\mathbb{C}$.

Lemma 3.3. Let $q \in [1,\infty[$ and g be an entire function on $\mathbb{C}$. Suppose that

$$\|g\|_{q,|\sigma|} < \infty$$

and

$$\left|g(z)\right| \le M\left(1+|z|\right)^m e^{a(\operatorname{Re} z)^2} \text{ for all } z \in \mathbb{C}$$

for some a, $M>0$ and $m \in \mathbb{N}$. Then $g=0$ on $\mathbb{C}$.

Proof. By (12)

$$\|g\|_{q,|\sigma|}^q \ge \int_{|\lambda|\ge 1} |g(\lambda)|^q \, d|\sigma|(\lambda)$$

$$= \int_{|\lambda|\ge 1} |g(\lambda)|^q \left|\frac{\lambda - ip}{\lambda}\right| \frac{d\lambda}{2\pi |c(|\lambda|)|^2}$$

$$\ge \int_{|\lambda|\ge 1} |g(\lambda)|^q \frac{d\lambda}{2\pi |c(|\lambda|)|^2}.$$

According to (13), there is a constant $k > 0$ such that $|c(|\lambda|)|^{-2} \ge k|\lambda|^{2\alpha+1}$ for all $\lambda \in \mathbb{R}$ with $|\lambda| \ge 1$. Therefore

$$\|g\|_{q,|\sigma|}^q \ge \frac{k}{2\pi} \int_{|\lambda|\ge 1} |g(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \ge \frac{k}{2\pi} \int_{|\lambda|\ge 1} |g(\lambda)|^q d\lambda,$$

which shows that $\|h\|_q < \infty$. The result is now a direct consequence of Lemma 3.2.

Lemma 3.4. Let $a, b > 0$, $d \ge 1, \gamma \in \mathbb{R}$ and

$$g(y) = \int_0^\infty e^{-a(x-by)^2} (1+x)^3 e^{\gamma x} dx, \quad y \ge 0.$$

Then there is a positive constant C such that

$$g(y) \le C(1+y)^d e^{\gamma b y} \text{ for all } y \ge 0.$$

Proof. By the convexity of x^d we have

$$g(y) = e^{\gamma b y} \int_{-by}^\infty e^{-az^2 + \gamma z} (1 + z + by)^d dz$$

$$\le e^{\gamma b y} \int_{-by}^\infty e^{-az^2 + |\gamma||z|} (1 + |z| + by)^d dz$$

$$\le e^{\gamma b y} \int_{-\infty}^\infty e^{-az^2 + |\gamma||z|} (1 + |z| + by)^d dz$$

$$= 2e^{\gamma b y} \int_{-\infty}^\infty e^{-az^2 + |\gamma| z} (1 + z + by)^d dz$$

$$\le \text{const.} e^{\gamma b y} \int_0^\infty e^{-az^2 + |\gamma| z} \left(1 + z^d + (by)^d\right) dz$$

$$= \text{const.} e^{\gamma b y} \left(\int_0^\infty e^{-az^2 + |\gamma| z} (1 + z^d) dz + (by)^d \int_0^\infty e^{-az^2 + |\gamma| z} dz \right)$$

$$\leq \text{const.}\left(1+y^{d}\right)e^{\gamma by}$$

$$\leq \text{const.}\left(1+y\right)^{d}e^{\gamma by}$$

which achieves the proof.

Lemma 3.5. Let $1 \leq q \leq \infty$ and $a>0$. Then there is a positive constant C such that for all $\lambda = \xi + i\eta \in \mathbb{R} + i\mathbb{R}$:

(i) $\left\|E_{1/4a}\Phi_{-\lambda}\right\|_{\infty} \leq C\left(1+|\lambda|\right)e^{\eta^2/4a}$;

(ii) $\left\|E_{1/4a}\Phi_{-\lambda}\right\|_{q,A} \leq C\left(1+|\lambda|\right)^{3} e^{\eta^2/4a+\frac{(2-q)\rho|\eta|}{2aq}}$; if $q<\infty$.

Proof. As the function $1/\sqrt{B(x)}$ is bounded, it follows from (8) and (16) that

$$\left|E_{1/4a}(x)\Phi_{-\lambda}(x)\right| \leq \text{const.}\left(1+|\lambda|\right)\left(1+|x|\right)^{2} e^{-ax^2+(|\eta|-\rho)|x|}$$

$$= \text{const.}\left(1+|\lambda|\right)\left(1+|x|\right)^{2} e^{\frac{\eta_2}{4a}} e^{-a\left(|x|-\frac{|\eta|}{2a}\right)^2-\rho|x|}$$

which proves (i). For $q<\infty$ we have

$$\left\|E_{1/4a}\Phi_{-\lambda}\right\|_{q,A} \leq \text{const.}\left(1+|\lambda|\right)e^{\eta^2/4a}\left(\int_0^{\infty} e^{-aq\left(x-\frac{|\eta|}{2a}\right)^2}\left(1+x\right)^{2q} e^{(2-q)\rho x}dx\right)^{1/q}$$

$$\leq \text{const.}\left(1+|\lambda|\right)\left(1+|\eta|\right)^{2} e^{\frac{\eta^2}{4a}+\frac{(2-q)\rho|\eta|}{2aq}}$$

$$\leq \text{const.}\left(1+|\lambda|\right)^{3} e^{\frac{\eta^2}{4a}+\frac{(2-q)\rho|\eta|}{2aq}}$$

by virtue of (10) and Lemma 3.4.

Lemma 3.6. Let $1 \leq p, p' \leq \infty$ such that $1/p + 1/p' = 1$. Let f be a measurable function on $\mathbb{R}$ such that $\left\|\mathbb{E}_{1/4a}^{-1} f\right\|_{p,A} < \infty$ for some $a>0$. Then the generalized Fourier transform of f is well defined and entire on $\mathbb{C}$. Moreover, there is a positive constant C such that for all $\lambda = \xi + i\eta \in \mathbb{R} + i\mathbb{R}$:

(i) $\left|\mathcal{F}_\Lambda(f)(\lambda)\right| \le C\left(1+|\lambda|\right)e^{\eta^2/4a}, if\ p=1;$

(ii) $\left|\mathcal{F}_\Lambda(f)(\lambda)\right| \le C\left(1+|\lambda|\right)^3 e^{\frac{\eta^2}{4a}+\frac{(2-p')\rho|\eta|}{2ap'}}, if\ p>1.$

Proof. The result follows easily by using Lemma 3.5, Hólder's inequality and the derivation theorem under the integral sign.

We can now state our main results.

Theorem 3.1. Let $1 \le p,q \le \infty$. Let f be a measurable function on $\mathbb{R}$ such that

$$E^{-1}_{1/4a} f \in L^p\left(\mathbb{R}, A(x)dx\right) \tag{19}$$

and

$$e^{b\lambda^2}\mathcal{F}_\Lambda(f) \in L^p\left(\mathbb{R},|\sigma|\right), \tag{20}$$

for some positive constants a and b. Then

If $ab>1/4$, we have $f=0$ almost everyhwhere.

If $ab>1/4$, for all $t \in \left]b,1/(4a)\right[$, E_t satisfies (19)-(20).

Proof. We divide the proof in two steps.

Step 1. $ab>1/4$

Let $t \in \left]1/(4a),b\right[$ and $g(\lambda)=e^{t\lambda^2}\mathcal{F}_\Lambda(f)(\lambda), \quad \lambda \in \mathbb{C}$.

By Lemma 3.6, g is entire on $\mathbb{C}$, and there is $C>0$ such that

$$\left|g(\lambda)\right| \le C\left(1+|\lambda|\right)^3 e^{t(\mathrm{Re}\,\lambda)^2}$$

for all $\lambda \in \mathbb{C}$. Furthermore,

$$\|g\|_{q,|\sigma|} = \left\|e^{b\lambda^2}\mathcal{F}_\Lambda(f)e^{(t-b)\lambda^2}\right\|_{q,|\sigma|} \le \left\|e^{b\lambda^2}\mathcal{F}_\Lambda(f)\right\|_{q,|\sigma|} < \infty.$$

(i) If $q<\infty$, it follows from Lemma 3.3 that $g(\lambda)=0$ for all $\lambda \in \mathbb{C}$. That is, $\mathcal{F}_\Lambda(f)(\lambda)=0$ for all $\lambda \in \mathbb{R}$. Therefore, $f=0$ a.e. on $\mathbb{R}$, by virtue of Corollary 2.1.

(ii) If $q=\infty$, then by Lemma 3.1 there is a constant $K \in \mathbb{C}$ such that $g(\lambda)=K$ for all $\lambda \in \mathbb{C}$. That is, $\mathcal{F}_\Lambda(f)(\lambda)=Ke^{-t\lambda^2}$ for all $\lambda \in \mathbb{R}$. Hence, $f=Ke^{t\rho^2}E_t$ a.e. on $\mathbb{R}$. But due to assumption (19), this is impossible unless $K=0$. Thus $f=0$ a.e. on $\mathbb{R}$.

Step 2. $ab<1/4$

Let $t \in \left]b,1/(4a)\right[$. [By (16), there are two positive constants $C_1(a,t)$ and $C_2(a,t)$ such that

$$C_1(a,t)e^{-\left(\frac{1}{4t}-a\right)x^2} \le E^{-1}_{\frac{1}{4a}}(x)E_t(x) \le C_2(a,t)e^{-\left(\frac{1}{4t}-a\right)x^2},$$

for all $x \in \mathbb{R}$. This shows that $E^{-1}_{\frac{1}{4a}} E_t \in L^p(\mathbb{R}, A(x)dx)$. Moreover,

$$\left\| e^{b\lambda^2} \mathcal{F}_\Lambda(E_t) \right\|_{q,|\sigma|} = e^{-t\rho^2} \left\| e^{-(t-b)\lambda^2} \right\|_{q,|\sigma|} < \infty$$

by virtue of (15) and the fact that σ is tempered. This ends the proof of Theorem 3.1.

Theorem 3.2. Let $1 \le p \le 2$ and $1 \le q \le \infty$. Let *f* be a measurable function on $\mathbb{R}$ satisfying (19) and (20) for some positive constants *a* and *b*. If $ab = 1/4$ then $f=0$ almost everywhere.

Proof. Let

$$g(\lambda) = e^{b\lambda^2} \mathcal{F}_\Lambda(f)(\lambda), \quad \lambda \in \mathbb{C}.$$

Let p' be the conjugate exponent of *p*. As by hypothesis $p' \ge 2$, we deduce from Lemma 3.6 that *g* is entire on $\mathbb{C}$, and there is $C>0$ such that

$$|g(\lambda)| \le C(1+|\lambda|)^3 e^{b(\mathrm{Re}\,\lambda)^2},$$

for all $\lambda \in \mathbb{C}$. The rest of the proof is now analogous to Step 1 in the proof of Theorem 3.1

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MODELLING THE ROLE OF ECOLOGY IN THE SPREAD OF CARRIER DEPENDENT INFECTIOUS DISEASES : AN SIS MODEL

By

Shikha Singh

UGC-Post Doctoral Fellow, Department of Mathematics and Statistics,
IIT Kanpur-208001, Uttar Pradesh, India
E-mail: shikha.ppn@gmail.com

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ABSTRACT

In this paper, an S-I-S epidemic model is investigated for carrier dependent infectious diseases such as measles, cholera etc. by considering explicitly the cumulative density of ecological factors, which are conducive to the growth of carrier population. It is assumed that the density of carrier population is governed by a generalized logistic model, the growth rate of which increases as the cumulative density of ecological factors increases. This density assumed to be human population density dependent. The model is analyzed by using qualitative theory of differential equations. It is shown that as the parameters governing the cumulative density of ecological factors increase, the number of infectives in the population increases. The model is discussed through the basic reproduction number and through numerical simulations also.

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Keywords and Phrases : Immigration; Logistic Model; Stability Analysis; Simulation; Carrier Dependent Infectious Diseases; Ecological Factors.

1. Introduction. There are many carrier dependent infectious diseases which afflict human populations mainly in the developing world. Various kinds of carriers such as flies, cockroaches, ticks, mites, etc., grow and survive in ecological habitats consisting of plants, bushes, parks in residential areas, water storage tanks, ponds etc. It may be pointed out that their population grow in the mounds formed by the roots of the bushes and plants as well as on the branches and the leaves and their populations increase further as the bushes and plants grow and become more dense. These carriers transport bacteria of cholera, dysentery, measles and gastroenteritis, etc. discharged by infectives of these diseases in the environment to susceptibles and thus spread these diseases in human population indirectly.

Ecological interactions can be complex with several factors often working together or in sequence [1],[8],[11], [18]-[20],[22], [23], [24],[26], [28], [35] e.g., the growth of suburban communities and the reforestation of the eastern United State have contributed to the rise in reported tick bites and infections and thus to the spread of Lyme disease in United States [3],[6],[14]. It is further noted that densities

of these ecological factors may increase due to various human population density related factors such as housing, agriculture and industrialization.

The modelling and analysis of various infectious diseases have been conducted by many researchers in the past as Bailey [2], Brauer [4], Brauer and Vanden Driessche [5], Chavez and Yakubu [7], Diekmann and Heesterbeek [9], Dietz and Eichner [10], Gao and Hethcote [12], Ghosh et.al. [13], Hethcote ([15]-[17]), Kalajdzievska and Li [21], May and Anderson [25], Murray [27], Safi et. al. [29], Singh et.al. ([30]-[32]), Zhang et.al. ([33-34]), However, it is noted that very little attention has been paid to the study of such diseases by considering effects of bacteria transporting carrier population, the density of which increases due to natural or human population density related ecological factors. Therefore, in this paper, an SIS model with constant immigration for a carrier dependent infectious diseases is proposed and analyzed by considering explicitly the effects of ecological factors which depends upon the density of human population. To be specific in the modelling process, it is assumed that the density of carrier population follows logistic model, the intrinsic growth rate and carrying capacity of which increase with the cumulative density of ecological factors. The cumulative density of ecological factors (such as biomass of leaves in bushes and plants etc.) is governed by a generalized logistic type equation, which depends upon human population density related factors.

This paper supplements the research work done by Singh et.al. [32] regarding the spread of carrier dependent infectious diseases with environmental effects.

2. An SIS Model with Ecological Effects. Let the total human population density $N(t)$ be divided into two sub classes: the susceptible density $X(t)$ and the infective density $Y(t)$, thus $N = X + Y$. Let $B(t)$ be the cumulative density of ecological factors (such as biomass of leaves in bushes, plants, etc.) favourable to the growth of carrier population and is governed by a generalized logistic equation. Let $C(t)$ be the carrier population density which is governed by a logistic model whose growth rate coefficient $s(B)$ and modified carrying capacity $L(B)$ increase as the cumulative density of ecological factors $B(t)$ increases. It is implicitly assumed here that the number of agents of the disease from the environment transported by carriers is proportional to their density.

Keeping in view the above considerations and by assuming simple mass action interaction, an SIS model is proposed as follows:

(2.1)
$$\frac{dX}{dt} = A - \beta XY - \lambda XC + \upsilon Y - dX,$$

$$\frac{DY}{dt} = \beta XY + \lambda XC - (\upsilon + \alpha + d)Y,$$

$$\frac{DN}{dt} = A - dN - \alpha Y,$$

$$\frac{dC}{dt} = s(B)C - \frac{s_0 C^2}{L(B)} - s_1 C,$$

$$\frac{dB}{dt} = r_0 B - (r_0 / K)B^2 - r_1 B + r_2 BN,$$

$$X + Y = N, X(0) > 0, Y(0) \geq 0, N(0) > 0, C(0) \geq 0, B(0) > 0.$$

For feasibility of the model (2.1), we must have $s(B) - s_1 > 0$ for $B \geq 0$ and $r_0 > r_1$. In model (2.1), A is constant immigration rate of human population, d is natural death rate constant, β and λ are the transmission coefficients due to infective and carrier populations respectively, α is the disease related deth rate constant, v is the recovery rate constant, s_1 is the deth rate coefficient of carriers due to natural factors as well as by control measures. Further, r_0 is the intrinsic growth rate coefficient of $B(t)$, r_1 is the natural depletion control rate of $B(t)$, r_2 is the growth rate coefficent of $B(t)$ due to human population densty related factors and K is the carrying capacity of $B(t)$ which is assumed to be constant.

In (2.1), $s(B)$ is the growth rate per capita of the carrier population density such that $s(B)-s_1$ is its intrinsic growth rate as compared to the usual logistic model. In view of the assumption that the growth rate capita $s(B)$ increases as the cumulative density of ecological factors $B(t)$ increases, we have

(2.2) $\quad s(0) = s_0 > 0$ and $\dfrac{ds(B)}{dB} \geq 0$,

where s_0 is the value of $s(B)$ when it is independent of B and $s_0 > s_1$.
Similarly, the fourth equation of model (2.1) governs the density of carrier population, when it is compared with usui logistic model, gives the carrying capacity as $\{L(B)/s_0\}\{s(B) - s_1\}$ where $L(B)$ is the usual carrying capacity of carrier population density. It is assumed that this modified carrying capacity, $L(B)$, increases as $B(t)$ increases, so we have,

(2.3) $\quad L(0) = L_0 > 0$ and $\dfrac{dL(B)}{dB} \geq 0$,

where L_0 is the value of $L(B)$ when is independent of B.
It is clear from the fourth equation of model (2.1), (2.2) and (2.3) that when $B=0$, the carrier population density increases in its natural environment and it tends to $L_0(1-(s_1/s_0))$ which may become zero as $s_1 \to s_0$.

3. Equilibrium Analysis. For analyzing the model (2.1), we consider the following reduced system (using $X + Y = N, Y < N, X > 0$):

(3.1) $\quad \dfrac{DY}{dt} = \beta(N - Y)Y + \lambda(N - Y)C - (v + \alpha + d)Y,$

$$\frac{DN}{dt} = A - dN - \alpha Y,$$

$$\frac{dC}{dt} = s(B)C - \frac{s_0 C^2}{L(B)} - s_1 C,$$

$$\frac{dB}{dt} = rB - \frac{r_0 B^2}{K} + r_2 BN,$$

where $r_0 - r_1 = r > 0$.

To analyze the above model, we first need the following lemma. The lemma establishes a region of attraction for the system (3.1).

Lemma 3.1: *The set*

$$\Omega = \{(Y, N, C, B) : 0 \le Y \le N \le A/d, 0 \le C \le C_m, 0 < B \le B_m\}$$

attracts all solutions initiating in the positive orthant,

where $B_m = (K/r_0)\{r + (r_2 A/d)\}$ and $C_m = \{L(B_m)/s_0\}\{s(B_m) - s_1\}$.

Proof. From the second equation of (3.1),we have

$$\frac{dN}{dt} = A - dN - aY \le A - dN \text{ which implies } 0 \le Y \le N \le A/d.$$

Further, from the last equation of model (3.1) we get

$$\frac{dB}{dt} \le \{r + (r_2 A)/d\}\left[B - \frac{B^2}{\{K/r_0\}\{r + (r_2 A)/d\}}\right],$$

which is a logistic type equation. Hence we have,

$$0 < B \le B_m \text{ where } B_m = \{K/r_0\}\{r + (r_2 A)/d\}.$$

Similarly, from the third equation of model (3.1), we get

$$\frac{dC}{dt} \le \{s(B_m) - s_1\}\left[C - \frac{C^2}{\{L(B_m)/s_0\}\{s(B_m) - s_1\}}\right],$$

which is the logistic type equation and gives:

$$0 \le C \le C_m \text{ where } C_m = \{L(B_m)/s_0\}\{s(B_m) - s_1\}; \text{ provided } s(B_m) > s_1.$$

The region of attraction, therefore is, as given by Ω defined avove.

It may be noted here that for the feasibility of the model, $s(B_m) > s_1$.

The results of equilibrium analysis are given in the following theorem:

Theorem 3.2: The system(3.1) has following six euqilibria:

(i) $E_0 = (0, A/d, 0, 0)$,

(ii) $E_1 = (0, A/d, 0, B_m)$, where $B_m = \{K/r_0\}\{r + (r_2 A)/d\}$,

(iii) $E_2 = (\bar{Y}, \bar{N}, 0, 0)$ where exists if $R = \dfrac{\beta A}{d(\upsilon+\alpha+d)} > 1$, where R are is the basic reproductive number, $\bar{Y} = \{\beta A - d(\upsilon+\alpha+d)\}/\beta(\alpha+d)$ and $\bar{N} = \{\beta A + \alpha(\upsilon+\alpha+d)\}/\beta(\alpha+d)$;

(iv) $E_3 = (\bar{Y}, \bar{N}, 0, \bar{B})$ which exists if $R > 1$,

where $\bar{B} = (K/r_0)(r + r_2\bar{N}), \bar{Y}, \bar{N}$ and R are defined above;

(v) $E_4 = (\tilde{Y}, \tilde{N}, \tilde{C}, 0)$ where

$$\tilde{Y} = \left[\tilde{\beta} + \sqrt{\tilde{\beta}^2 + 4\beta(1+(\alpha/d))(\lambda A\tilde{C}/d)}\right] / \{2\beta(\alpha+d)/d\},$$

$$\tilde{\beta} = \{(\beta A/d) - (\upsilon+\alpha+d) - \lambda\tilde{C}(1+(\alpha/d))\},$$

$$\tilde{C} = (L_0/s_0)(s_0 - s_1) \text{ and } \tilde{N} = (A - \alpha\tilde{Y})/d;$$

(vi) $E_5 = (Y^*, N^*, C^*, B^*)$ the endemic equilibrium, which exists in the subregion $\Omega_s = \{(Y,N,C,B) : 0 \le Y \le A/d, A/(\alpha+d) < N < A/d, 0 \le C \le C_m, 0 < B \le B_m\}$ of Ω, provided the basic reproductive number, $R = \dfrac{\beta A}{d(\upsilon+\alpha+d)} < 1$. It is noted here that if the equilibrium point E_5 exists then E_2 and E_3 will not exist.

Proof. The proof of existence of E_0, E_1, E_2, E_3 or E_4 is trivial. The endemic equilibrium point E_5 is given by the solution of the following set of equations obtained from (3.1) by putting left hand sides to zero:

(3.2) $\beta Y^2 + Y\{(\upsilon+\alpha+d) - \beta N + \lambda C\} - \lambda NC = 0,$

(3.3) $Y = (A - dN)/\alpha,$

(3.4) $C = \{s(B) - s_1\}\{L(B)/s_0\},$

(3.5) $B = (K/r_0)(r + r_2 N).$

Eliminating Y from (3.2) and (3.3), we get the following relalion:

(3.6) $F(N) = (\beta/\alpha^2)(A - dN)^2 + \{(A - dN)/\alpha\}\{(\upsilon+\alpha+d) - \beta N + \lambda C\} - \lambda NC = 0,$

where C can be obtained in terms of N by equations (3.4) and (3.5).

It is noted from (3.6) that $F(A/(\alpha+d))$ is positive provided $R = \dfrac{\beta A}{d(\upsilon+\alpha+d)} < 1$ and $F(A/d)$ is negative, bacause $s(B_m) - s_1 > 0$.

Thus, it is clear that there exists a root N^* of $F(N) = 0$ in $A/(\alpha+d) < N < A/d$

provided $R<1$. Further, this root N^* will be unique if

(3.7) $$\frac{dF(N)}{dN}<0 \text{ for } A/(\alpha+d)<N<A/d.$$

To show this, we differentiate (3.6) to get

(3.8) $$\frac{dF(N)}{dN}=-\left(\frac{2\beta d}{\alpha^2}\right)(A-dN)-\left(\frac{d}{\alpha}\right)\{(\upsilon+\alpha+d)-\beta N+\lambda C\}-\left(\frac{\beta}{\alpha}\right)(A-dN)$$
$$-\lambda C-\frac{\lambda K r_2}{s_0 r_0}\left[\{s(B)-s_1\}L'(B)+s'(B)L(B)\right]\left(\frac{N(\alpha+d)-A}{\alpha}\right).$$

It can be easily seen from (3.8) that $\frac{dF(N)}{dN}<0$ provided $R=\frac{\beta A}{d(\upsilon+\alpha+d)}<1$ in the region $A/(\alpha+d)<N<A/d$.

Now, knowing the value of N^*, the values of Y^*, C^* and B^* can be uniquely determined from (3.3), (3.4) and (3.5). Hence, E_5 exists in Ω_s provided $R<1$.

Remarks.

(i) We note that

$$\left.\frac{dF(N)}{dN}\right|_{E5}=-(d/\alpha)\{\beta Y^*+(\lambda N^*C^*/Y^*)\}-\beta Y^*-\lambda C^*$$
$$-(\lambda K r_2/s_0 r_0)\left[s'(B^*)L(B^*)+\{s(B^*)-s_1\}L'(B^*)\right](N^*-Y^*)<0.$$

Thus, it is clear that the above condition (3.7) is automatically satisfied at the equilibrium point E_5.

(ii) We note that $\left.\frac{dY}{dr}\right|_{E_5}>0$ and $\left.\frac{dC}{dr}\right|_{E_5}>0$.

On substituting the value of N from (3.3) into (3.2) and (3.5), we get

$$\beta Y^2+Y\left[(\upsilon+\alpha+d)-\beta\{(A-\alpha Y)/d\}+\lambda C\right]-\lambda\{(A-\alpha Y)/d\}C=0$$

and $$B=(Kr/r_0)+(r_2K/r_0)\{(A-\alpha Y)/d\}.$$

On differentiating the above equations w.r.t. 'r' and on simplification, we get

(3.9) $$\frac{dY}{dr}\left[\beta Y+(\alpha\beta/d)Y+(\lambda NC/Y)+(\alpha\lambda C/d)\right]=\lambda(N-Y)\frac{dC}{dr} \text{ and}$$

(3.10) $$\frac{dB}{dr}=(K/r_0)-(\alpha r_2 K/r_0 d)\frac{dY}{dr}.$$

Now from (3.4), we have

$$(3.11)\quad \frac{dC}{dB} = I(B)/s_0,$$

where $I(B) = \{s(B) - s_1\}L'(B) + L(B)s'(B)$.

Further, we note that

$$(3.12)\quad \frac{dC}{dr} = \frac{dC}{dB}\frac{dB}{dr}.$$

Using (3.9)-(3.12), we get

$$(3.13)\ \left.\frac{dY}{dr}\right|_{E_5} = \frac{\lambda(N^* - Y^*)(K/s_0 r_0)I(B^*)}{BY^* + (\lambda N^* C^*/Y^*) + (\alpha\beta/d)Y^* + (\lambda C^* \alpha/d) + \dfrac{\lambda(N^* - Y^*)K\alpha r_2}{s_0 r_0 d} I(B^*)} > 0.$$

Thus it is seen here that as the growth rate coefficient of cumulative density of ecological factors increases, the infective human population density increases at the equilibrium E_5.

Further ,on substituting the value of $\left.\frac{dY}{dr}\right|_{E_5}$ from (3.13) into (3.9), we get

$$\left.\frac{dC}{dr}\right|_{E_5} = \frac{K\left[\beta Y^* + (\lambda N^* C^*/Y^*) + (\alpha\beta/d)Y^* + (\lambda C^* \alpha/d)\right]I(B^*)}{s_0 r_0\left[\beta Y^* + (\lambda N^* C^*/Y^*) + (\alpha\beta/d)Y^* + (\lambda C^* \alpha/d) + \left\{\dfrac{\lambda(N^* - Y^*)K\alpha r_2}{s_0 r_0 d}\right\} I(B^*)\right]} > 0,$$

which shows that as the growth rate coefficient of cumulative density of ecological factors increases ,the carrier population density increases at the equilibrium E_5. In the similar manner, we can show that

$$\textbf{(iii)}\ \left.\frac{dY}{dr_2}\right|_{E_5} = \frac{\lambda(N^* - Y^*)(KN^*/s_0 r_0)I(B^*)}{\beta Y^* + (\lambda N^* C^*/Y^*) + (\alpha\beta/d)Y^* + (\alpha\lambda C^*/d) + \dfrac{\lambda(N^* - Y^*)\alpha K r_2 I(B^*)}{s_0 r_0 d}} > 0.$$

Thus, it is clear that as the growth rate coefficient of cumulative density of ecological factors due to human population density related factors increases, the infective human population density increases at the equilibrium E_5.

(iv) Also, we have

$$\left.\frac{dC}{dr_2}\right|_{E_5} = \frac{(KN^*/s_0 r_0)\left[\beta Y^* + (\lambda N^* C^*/Y^*) + (\alpha\beta/d)Y^* + (\lambda C^* \alpha/d)\right]I(B^*)}{\left[\beta Y^* + (\lambda N^* C^*/Y^*) + (\alpha\beta/d)Y^* + (\lambda C^* \alpha/d) + \left\{\dfrac{\lambda(N^* - Y^*)K\alpha r_2}{s_0 r_0 d}\right\} I(B^*)\right]} > 0$$

which shows that as the growth rate cofficient of cumulative density of ecological factors due to human population density related factors increases, the carrier population density increses at the equilibrium E_5.

4. Stability Analysis. In the following, the local stability results of equilibria E_0, E_1, E_2, E_3, E_4 and E_5 are stated,the details of which are given in the Appendtx *A*.

Theorem 4.1: the equilibira E_0, E_1, E_2, E_3 and E_4 are locally unstable and the eqilibrium E_5, if exists, is locally asymptutically stable provided the following conditions are satisfied,

$$(4.1)\quad \alpha\lambda^2 C^{*2} < 2d\beta^2 Y^{*2}/3,$$

$$(4.2)\quad 3\alpha\lambda^2 (N^*-Y^*)^2 L^2(B^*)\left\{s'(B^*)+\frac{s_0 C^*}{L^2(B^*)}L'(B^*)\right\}^2 K^2 r_2^2 < d\beta^2 s_0^2 Y^{*2} r_0^2 .$$

Remrak. It is pointed out here that the inequality (4.2) is automatically when $s(B)$ and $L(B)$ are independent of B i.e. when the density of carrier population is not affected by the cumulative density of ecological factors. This shows that ecological factors have destabilizing effects the system.

The global stability result of E_5 is given the following theorem:

Theorem 4.3. In addition to assumptions (2.2) and (2.3), let $S(B)$ and $L(B)$ satisfy $0 \le s'(B) \le p$ and $0 \le L'(B) \le q$ for some positive constants p and q in Ω_s, then E_5, if exists, is globally asymptotically stable in Ω_s provided the following inequalities are staisfied:

$$(4.3)\quad \alpha\lambda^2 C_m^2 < 2d\beta^2 Y^{*2}/3,$$

$$(4.4)\quad 3\alpha\lambda^2 (N^*-Y^*)^2 L^2(B^*)\left\{p+s_0\frac{C_m q}{L_0^2}\right\}^2 K^2 r_2^2 < d\beta^2 s_0^2 Y^{*2} r_0^2 .$$

Remark. It is noted here that if $p=q=0$, the inequality (4.4) is automatically satisfied which shows that the cumulative density of ecological factors have destabilizing effects on the system.

Proof. See Appenix B.

The above theorem imply that under certain conditions if the density of carrier population increases, then the number of infectives in human population increases leading to increased spread of the carrier dependent infectious diseases. These theorems also suggest that such an infectious disease become more endemic due to immigration.

5. Simulation. In this section we analyze the model (3.1) for appropriate vaules of parameters to show the global stability behavior of E_5.

For numerical simulation we choose $s(B)$ and $L(B)$ as follows:

(5.1) $s(B) = s_0 + s_{11}B$ and $L(B) = L_0 + L_{11}B$.

In the following ,we shall first show the existence equilibrium point and then its global asymptotical stability behavior by taking the following values of the vari ous parameters:

$\beta = 5.1\times10^{-7}, \upsilon = 0.012, \alpha = 0.0005, d = 0.0004, \lambda = 2.1\times10^{-8}, A = 10, s_0 = 0.9, s_1 = 0.6,$

$s_{11} = 2.0\times10^{-6}, L_0 = 100000, L_{11} = 4\times10^{-6}, r_0 = 0.9, r = 0.7, r_2 = 0.225\times10^{-5}, K = 26000.$

The values of N* and Y* are determined as the point of intersection of the following equations (which are obtained by eliminating C and B from (3.2) to (3.5)):

$$(5.2)\ \beta Y^2 + Y\Big[(\upsilon+\alpha+d) - \beta N + \lambda\big\{(s_0 - s_1 + s_{11}(K/r_0)(r + r_2N))(L_0 + L_{11}(K/r_0)(r + r_2N))/s_0\big\}\Big]$$

$$-\lambda N\big\{(s_0 - s_1 + s_{11}(K/r_0)(r + r_2N))(L_0 + L_{11}(K/r_0)(r + r_2N))/s_0\big\} = 0$$

and

(5.3) $Y = (A - dN)/\alpha$.

Using the above numerical values of various parameters, these equations are plotted in Y–N plane by using MATLAB and are shown in following Fig.1:

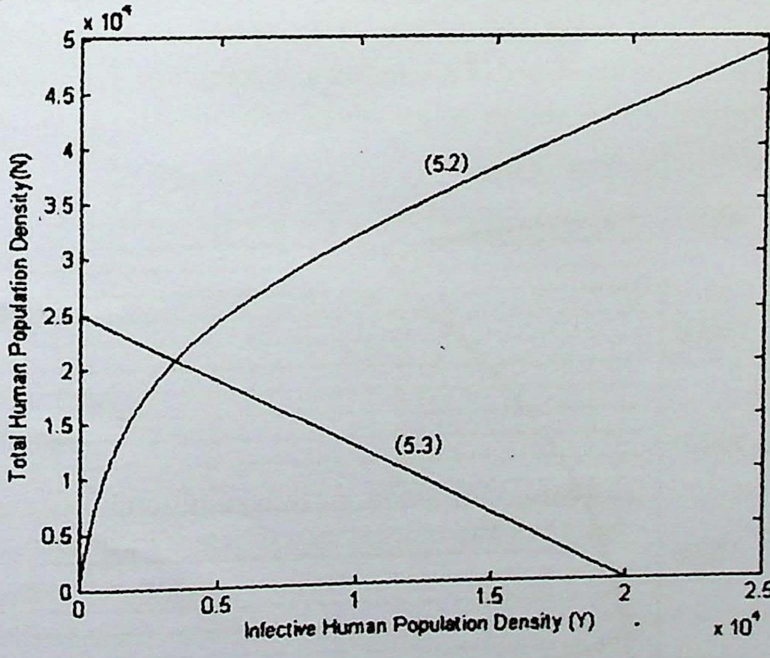


Fig. 1 : Existence of (Y^*, N^*)

Here, the point of intersection in the first quadrant gives $Y^* = 3411.91$,

$N^* = 20735.10$ and the values of C^* and B^* can be obtained from the equations (3.4) and (3.5) respectively.

Further, the endemic equilibrium point E_5 can be obtained directly by numerically solving the equations (3.2) to (3.5). Thus, solving these equations along with (5.1) (with MAPLE) and using the numerical values of various parameters given above, we get

$Y^* = 3411.912520, N^* = 20735.10935, C^* = 38126.70052, B^* = 21570.00433$.

Hence the existence of the endemic equilibrium point is shown for the given set of parameters. The variational matrix corresponding to this equilibrium E_5, is obtained as,

$$M_5 = \begin{bmatrix} -0.006602 & 0.002532 & 0.000363 & 0 \\ -0.0005 & -0.0004 & 0 & 0 \\ 0 & 0 & -0.34200 & 0.07600 \\ 0 & 0.048370 & 0 & -0.74400 \end{bmatrix}.$$

The eigenvalues of this matrix are:–0.0064, –0.0006, –0.7440, –0.3420 which are all negative. Hence E_5 is locally stable.

It is also pointed out here that for the above set of parameters, the conditions for local stability (4.1,4.2) and global stability (4.3,4.4) have been checked and they are satisfied. Also for the above set of parameters, a computer generated graph of Y vs N for the model (3.1) is shown below which indicates the global stability of the point (Y^*, N^*) in the Y–N plane :

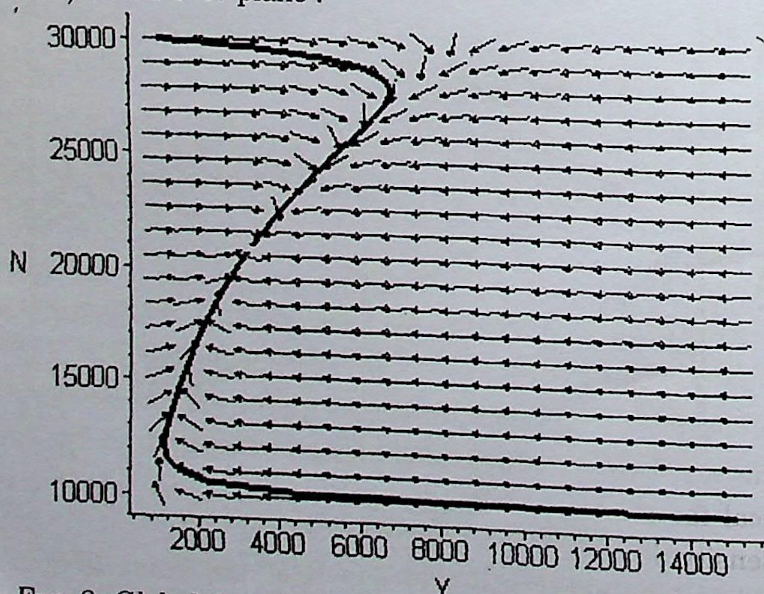


Fig. 3: Global Stability of Equilibrium Point (Y^*, N^*)

Further, variation of infective human population (Y) with time (t) has been shown for different values of parameters r and r_2 in Figs. 3 and 4.

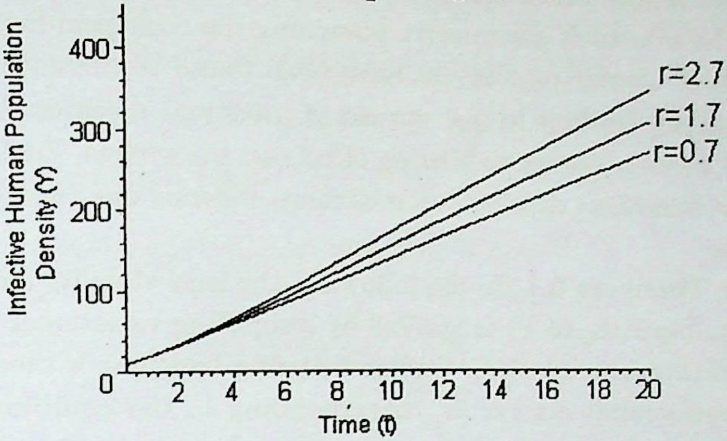


Fig. 3. Variation of Y with t for different values of parameter

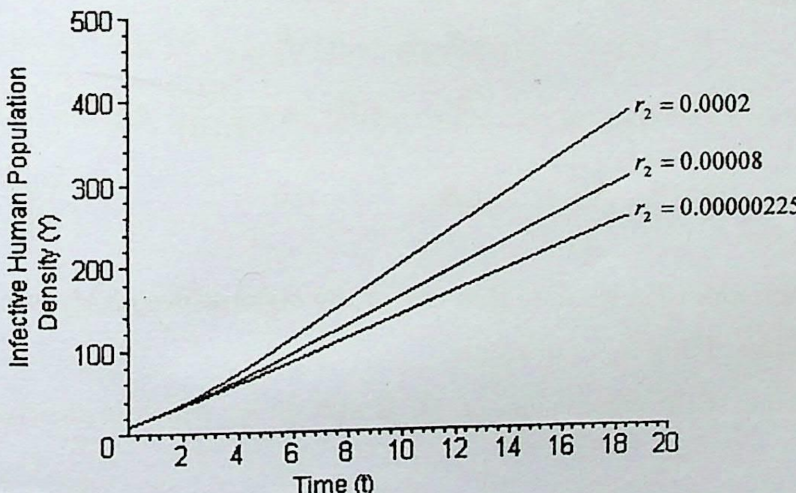


Fig.4. Variation of Y with t for different values of parameter r_2

From above figures, it can be seen that Y increases as r or r_2 increases. Thus, it is clear that the infective human population density increases as the growth rate coefficient of cumulative density of ecological factors and growth rate coefficient of the same due to human population density related factors, increase.

6. Conclusions. In this paper, we have proposed and analyzed as SIS model for a carrier dependent infectious disease by considering ecological effects. We have assumed only those ecological factors which are conducive to the growth of density of the carrier population. It is assumed that the cumulative density of such ecological factors is governed by a generalized logistic equation which is population density dependent. The equation governing the carrier population has been assumed to be a genralized logistic model, the intrinsic growth rate and

carrying capacity of which increase as the cumulative density of ecological factors increases. The model has been analyzed analytically as well as by computer simulation. The effects of parameters governing the ecological factors conductive to the growth of carrier population have been found to increase the density of carrier population, leading to fast spread of infectious diseases. It is noted that the ecological factors have destabilizing effects on the system. Further it has been found that an infectious disease becomes more endemic due to immigration.

Appendix A.

Proof of the Theorem 4.1. In the following, the local stability behaviour of each of the five equilibria E_0 to E_4 is studied by computing variational matrix and the endemic equilibrium point E_5 is studied by using Lyapunov's theory.

The variational matrix M_i corresponding to the equilibrium points E_i, $i=1,2,3,4,5$ is given by

$$M_i = \begin{bmatrix} \beta N - 2\beta Y - \lambda C - (v+\alpha+d) & \beta Y + \lambda C & \lambda(N-Y) & 0 \\ -\alpha & -d & 0 & 0 \\ 0 & 0 & s(B) - s_1 - \dfrac{2s_0 C}{L(B)} & s'(B)C + \dfrac{s_0 C^2 L'(B)}{L^2(B)} \\ 0 & r_2 B & 0 & r - \dfrac{2r_0 B}{K} + r_2 N \end{bmatrix}.$$

For equilibrium point E_0, we note that one of the eigenvalues of M_0 is $r+(r_2 A/d)$, which is positive. Thus E_0 is unstable.

We note that one of the eigenvalues of M_1 is $s(B_m)-s_1$, which is positive. Thus E_1 is unstable.

We find that one of the eigenvalues of M_2 is $r+r_2\bar{N}$, which is positive. Thus E_2 is unstable.

We find that one of the eigenvalues of E_2 is $s(\bar{B})-s_1$, which is positive. Thus E_3 is unstable.

For equilibrium point E_4, we find that one of the eigenvalues of M_4 is $r+r_2\tilde{N}$, which is positive. Thus, E_4 is unstable.

Stability Behaviour of $E_5 = (Y^*, N^*, C^*, B^*)$.

Since the nature of E_5 cannot be seen easily from the variational matrix, we study it's local stability by using Lyapunov's theory. For this, we use following

positive definite function

(A.1) $V=(k_0/2)(Y-Y^*)^2+(k_1/2)(N-N^*)^2+(k_2/2)(C-C^*)^2+(k_3/2)(B-B^*)^2$.

Thus, dV/dt along the lineraized system (3.1) can be written after rearrangement of terms as :

$$\frac{dV}{dt}=(k_0\beta Y^*-k_1\alpha)yn-(k_0\lambda C^*N^*/Y^*)y^2-(k_0\beta Y^*/3)y^2$$

$$-\left[(k_0\beta Y^*/3)y^2-(k_0\lambda C^*)yn+(k_1d/2)n^2\right]$$

$$-\left[(k_0\beta Y^*/3)y^2-\{k_0\lambda(N^*-Y^*)\}yc+\left\{\frac{k_2s_0C^*}{2L(B^*)}\right\}c^2\right]$$

$$-\left[\left\{\frac{k_2s_0C^*}{2L(B^*)}\right\}c^2-k_2C^*\left\{s'(B^*)+\frac{s_0C^*}{L^2(B^*)}L'(B^*)\right\}cb+(k_3r_0B^*/2K)b^2\right]$$

$$-\left[(k_3r_0B^*/2K)b^2-(k_3r_2B^*)nb+(k_1d/2)n^2\right].$$

C h $k_1=1$ and $k_0=(\alpha/\beta Y^*)$, we get following inequalities for $\dot{V}$ to be negative definite,

(A.2) $\alpha\lambda^2C^{*2}<2d\beta^2Y^{*2}/3$,

(A.3) $$\frac{3}{2}\frac{\alpha\lambda^2L(B^*)(N^*-Y^*)^2}{s_0\beta^2Y_2^{*2}C^*}<k_2<\frac{s_0(r_0B^*/K)}{L(B^*)C^*\left\{s'(B^*)+\frac{s_0C^*}{L^2(B^*)}L'(B^*)\right\}^2}k_3,$$

(A.4) $$k_3<\frac{r_0d}{Kr_2^2B^*}.$$

Now, by choosing $k_3=r_0d/2Kr_2^2B^*$, the inequality (A.4) stisfied automatically. Further, we can choose k_2 satisfying (A.3) provided that

$$3\alpha\lambda^2(N^*-Y^*)^2L^2(B^*)\left\{s'(B^*)+\frac{s_0C^*}{L^2(B^*)}L'(B^*)\right\}^2K^2r_2^2<ds_0^2\beta^2Y^{*2}r_0^2 \text{ is satisfied.}$$

It can be seen that $\dot{V}$ is negative definite under the conditions (4.1) and (4.2) as stated in Theorem 4.1. Hence the result.

Appendix B

Proof of the Theorem 4.3. We prove the theorem by using the following positive

definite function

(B.1) $$v = k_0\left(Y - Y^* - Y^* \text{ in } Y/Y^*\right) + \frac{k_1}{2}\left(N - N^*\right)^2 + k_2\left(C - C^* - C^* \text{ in } C/C^*\right)$$

$$+k_1\left(B - B^* - B^* \text{ in } B/B^*\right).$$

Differentiating (B.1) and using (3.1), we get

(B.2) $$\frac{dV}{dt} = (k_0\beta - k_1\alpha)(Y - Y^*)(N - N^*) - (k_0\lambda CN/YY^*)(Y - Y^*)^2 - (k_0\beta/3)(Y - Y^*)^2$$

$$-\left[(k_0\beta/3)(Y - Y^*)^2 - (k_0\lambda C/Y^*)(Y - Y^*)(N - N^*) + (k_1 d/2)(N - N^*)^2\right]$$

$$-\left[\left(\frac{k_0\beta}{3}\right)(Y - Y^*)^2 - (k_0\lambda/Y^*)(N^* - Y^*)(Y - Y^*)(C - C^*) + \frac{k_2 s_0}{2L(B^*)}(C - C^*)^2\right]$$

$$-\left[\frac{k_2 s_0}{2L(B^*)}(C - C^*)^2 - k_2\left\{f(B) + s_0 C g(B)\right\}(C - C^*)(B - B^*) + \left(\frac{k_3 r_0}{2K}\right)(B - B^*)^2\right]$$

$$-\left[(k_1 d/2)(N - N^*)^2 - k_3 r_2(N - N^*)(B - B^*) + (k_3 r_0/2K)(B - B^*)^2\right].$$

where $f(B)$ and $g(B)$ are defined as

(B.3) $$f(B) = \begin{cases} \dfrac{s(B) - s(B^*)}{B - B^*}, & B \neq B^* \\ \dfrac{ds}{dB}, & B = B^* \end{cases}$$

and

(B.4) $$g(B) = \begin{cases} \dfrac{L(B) - L(B^*)}{B - B^*}\dfrac{1}{L(B)L(B^*)}, & B \neq B^* \\ \dfrac{1}{L^2(B)}\dfrac{dL}{dB}, & B = B^*. \end{cases}$$

As stated in the theorem, $s(B)$ and $L(B)$ satisfy the following conditions :

(B.5) $0 \le s'(B) \le p$ and $0 \le L'(B) \le q$ for some positive constants p and q.

Hence, from (B.3), (B.4) (B.5) and using the mean value theorem, we have

(B.6) $|f(B)| \le p$ and $|g(B)| \le q/L_0^2$.

Choosing $k_1 = 1$ and $k_0 = \alpha/\beta$, we have following inequalities for $\dot{V}$ to be negative definite

(B.7) $\alpha\lambda^2 C_m^2 < 2d\beta^2 Y^{*2}/3$,

(B.8) $\frac{3}{2}\frac{\alpha\lambda^2 L(B^*)(N^*-Y^*)^2}{s_0\beta^2 Y^{*2}} < k_2 < \frac{r_0 s_0}{KL(B^*)\{p + s_0 C_m q/L_0^2\}^2} k_3$,

(B.9) $k_3 < r_0 d/(Kr_2^2)$.

By choosing $k_3 = r_0 d/2Kr_2^2$, the inequality (B.9) is automatically satisfied. So, we can choose k_2 satisfying (B.8) provided that

$$3\alpha\lambda^2 L^2(B^*)(N^*-Y^*)^2 \left\{p + \frac{s_0 C_m q}{L_0^2}\right\}^2 K^2 r_2^2 < r_0^2 d\beta^2 s_0^2 Y^{*2}$$ is satisfied.

Hence, E_5 is globally asymptotically stable if (4.3) and (4.4) are satisfied as stated in the Theorem 4.3. Hence the result.

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SOLUTION OF CERTAIN INTEGRAL EQUATION INVOLVING *I*-FUNCTION

By

Vandana Jat

Department of Mathematics, Patel College of Sciences and Technology Ratibad, Bhopal-462003, Madhya Pradesh, India

and

V.P. Saxena

Sagar Institute of Research Technology of Science, Bhopal-462041, Madhya Pradesh, India

email :vinodpsaxena@gmail.com

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ABSTRACT

In this paper we have obtained solution of certain new a integral equation in which Saxena's *I*-function occurs as a part of the kernel. Erdélyi-Kober fractional integral operators have been applied to transform the kernel in desired form, so that the conditions of unsymmetric fourier kernels are satisfied. These results may be useful in many applications of integral equations. The special cases are obtained from the known results. First we shall transform the given equation in a desired form, by making use of fractional integral operators. Then the integral equation thus obtained, will be split into a pair of equations, with the help of Laplace transforms. The resulting integral equations will, then be solved by using the formulas, already derived.

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Key Words and Phrases : Saxena's *I*-function, Erdélyi-Kober fractional integral operators, Integral equation, Laplace transforms.

1. Introduction. Solution of various generalized sets of dual and triple integral equations have been given by Charles Fox ([1],[2]), R.K. Saxena [5] and V.P. Saxena [6]. In each case the responsibility of final solution goes to integral equation, having *H*-function as the kernel of the type corresponding to the set. But due to non-availability of much inversion formula the kernels are more or less restricted to some standard forms.

In this paper we shall give solution to the integral equation having considerably more generalized kernel. We shall use Laplace and Generalized Laplace transforms of Meijer [4] and operators of fractional integration. The generalization of Fox's *H*-functions, called, *I*-function is given by Saxena [7], its definition is given as

$$(1.1)\quad I(Z)=I^{m,n}_{p_i,q_i;r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_j,n;(a_{ji},\alpha_{ji})_{n+1,pi}\\(b_j,\beta_j)_1;(a_{ji},\beta_{ji})_{m+1,qi}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\phi(\xi)z^{\xi}d\xi,$$

$$\phi(\xi)=\frac{\prod_{j=1}^{m}\sqrt{(b_j-\beta_j\xi)}\prod_{j=1}^{n}\sqrt{(1-a_j+\alpha_j\xi)}}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\sqrt{(1-b_{ji}+\beta_{ji}\xi)}\prod_{j=n+1}^{p_i}\sqrt{(a_{ji}-\alpha_{ji}\xi)}\right\}},$$

with the conditions of existence given in (2.17).

(i) If we put $r=1$ in (1.1), it reduces to Fox's H-function

$$I^{m,n}_{p_i,q_i;1}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,pi}\\(b_j,\beta_j)_{1,m};(a_{ji},\beta_{ji})_{m+1,qi}\end{matrix}\right.\right]=H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m};(b_j,\beta_j)_{m+1,q}\end{matrix}\right.\right],$$

where $p_1=p, q_1=q, a_{ji}=a_j, b_{ji}=b_j, \alpha_{ji}=\alpha_j$, and $\beta_{ji}=\beta_j$.

(ii) If we put $\alpha_j=\beta_j=\alpha_{ji}=\beta_{ji}=1$ in (1.1), it reduces to a function I_g defined as

$$I_g=\frac{1}{2\pi i}\int_L\eta(\xi)z^{\xi}d\xi,$$

where

$$\eta(\xi)=\frac{\left\{\prod_{j=1}^{m}\Gamma(b_j-\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\xi)\right\}}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{qi}\Gamma(1-b_{ji}+\xi)\prod_{j=n+1}^{pi}\Gamma(a_j-\xi)\right\}}.$$

(iii) If we put $r=1,\alpha_j=\beta_j=\alpha_{ji}=\beta_{ji}=\alpha$ in (2.1), it reduces to a Meijer's G-function, i.e.

$$I^{m,n}_{p_i,q_i;1}\left[z\left|\begin{matrix}(a_j,\alpha)_{1,n};(a_{ji},\alpha)_{n+1,pi}\\(b_j,\beta)_{1,m};(b_{ji},\alpha)_{m+1,qi};\end{matrix}\right.\right]=\frac{1}{\alpha}G^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j,1)_{1,n};(a_j,1)_{n+1,p}\\(b_j,1)_{1,m};(b_{,},1)_{m+1,q}\end{matrix}\right.\right].$$

2. Problem and Solution. We consider the integral equation

$$g(x)=\int_0^\infty I\left[ux\left|\begin{matrix}\left\{\left(1-a_j,e_j\right)_{1,q}\right\};\left\{\left(a_{ji},e_{ji}\right)_{q+1,t_i}\right\}\\ \left\{\left(b_j,f\right)_{1,p}\right\};\left\{\left(1-b_{ji},f_{ji}\right)_{p+1,s_i}\right\}\end{matrix}\right.\right]F(u)du \tag{2.1}$$

Here $g(x)$ is known and $f(x)$ is to be determined. The notation of the I-function, used here in different from (1.1).

Mellin transform of I-function is given by

$$M\left[I\left[u\left|\begin{matrix}\left\{\left(1-a_j,e_j\right)_{1,q}\right\};\left\{\left(a_{ji},e_{ji}\right)_{q+1,t_i}\right\}\\ \left\{\left(b_j,f\right)_{1,p}\right\};\left\{\left(1-b_{ji},f_{ji}\right)_{p+1,s_i}\right\}\end{matrix}\right.\right]\right]=\frac{\prod_{j=1}^{p}\Gamma\left(b_j-f_j\xi\right)\prod_{j=1}^{q}\Gamma\left(1-a_j+e_j\xi\right)}{\sum_{i=1}^{r}\left\{\prod_{j=p+1}^{s_i}\Gamma\left(1-b_{ji}-f_{ji}\xi\right)\prod_{j=q+1}^{t_i}\Gamma\left(a_{ji}-e_{ji}\xi\right)\right\}}. \tag{2.2}$$

The Mellin transform of a function $f(x)$ is defined as

$$F(\xi)=M\left[f(u)\right]=\int_0^\infty u^{\xi-1}f(u)du. \tag{2.3}$$

The Parseval's theorem of the transform is

$$\int_0^\infty g(ux)f(u)du=\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}x^{-\xi}g(\xi)f(1-\xi)d\xi. \tag{2.4}$$

From the equation (2.4) and (2.2), we obtain

$$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{p}\Gamma\left(b_j-f_j\xi\right)\prod_{j=1}^{q}\Gamma\left(1-a_j-e_j\xi\right)}{\sum_{i=1}^{r}\left\{\prod_{j=p+1}^{s_i}\Gamma\left(1-b_j-f_j\xi\right)\prod_{j=q+1}^{t_i}\Gamma\left(a_j+e_j\xi\right)\right\}}x^{-\xi}F(1-\xi)d\xi=g(x), \tag{2.5}$$

where $F(\xi)=M\left[f(u)\right]$.

Now we shall use following definition of Beta function

$$\int_x^\infty\left(v^{1/c}-x^{1/c}\right)^{b-\beta-1}v^{1/c-b/c-s-1}dv=c\Gamma(b-\beta)x^{\beta/c-s}\frac{\Gamma(\beta+cs)}{\Gamma(b+cs)}. \tag{2.6}$$

Replace x by v in (2.6), multiply by $\left(v^{1/f_1}-x^{1/f_1}\right)^{b_1-\mu_1-1}v^{1/f_1-b/f_1-1}$,

and integrate under the integral sign with respect to v form x to ∞ where $x>1$ to obtain

$$\int_x^{\infty}\left(v^{1/f_1}-x^{1/f_1}\right)^{b_1-\mu_1-1}v^{1/f_1-b_1/f_1-1}dv=\frac{f_1\Gamma(b_1-\mu_1)\Gamma(\mu_1+f_1\xi)}{\Gamma(b_1+f_1\xi)}x^{-\mu_1/f_1}.$$

Accordingly from eq. (2.5), we arrive at

$$(2.7)\quad \frac{1}{2\pi i}\int_L \frac{\prod_{j=2}^{p}\Gamma(b_j+f_j\xi)\Gamma(b_1+f_1\xi)\Gamma(\mu_1+f_1\xi)\prod_{j=1}^{q}\Gamma(1-a_j-e_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=p+1}^{s_i}\Gamma(1-b_{ji}-f_{ji}\xi)\prod_{j=q+1}^{t_i}\Gamma(a_{ji}+e_{ji}\xi)\right\}}\frac{1}{\Gamma(b_1+f_1\xi)}x^{-\xi}f(1-\xi)d\xi$$

$$=\frac{1}{f_1\Gamma(b_1-\mu_1)}x^{\mu_1/f_1}\int_x^{\infty}\left(v^{1/f_1}-x^{1/f_1}\right)^{b_1-\mu_1-1}v^{1/f_1-b_1/f_1-1}g(v)dv.$$

Let us denote right hand side of above equations by $g_1(x)$. Then it is evident that

$$(2.8)\quad g_1(x)=\Re[b_1-\mu_1,\mu_1/f_1:1/f_1]g(x),$$

where $\Re$ is second operator of fractional integration defined in (2.6).
We apply the operator again, so that we get

$$(2.9)\quad \frac{1}{2\pi i}\int_L \frac{\prod_{j=3}^{p}\Gamma(b_j+f_j\xi)\Gamma(\mu_1+f_1\xi)\Gamma(\mu_2+f_2\xi)\prod_{j=1}^{q}\Gamma(1-a_j-e_j\xi)}{\sum_{i=1}^{r}\left[\prod_{j=p+1}^{s_i}\Gamma(1-b_{ji}-f_{ji}\xi)\prod_{j=q+1}^{t_i}\Gamma(a_{ji}+e_{ji}\xi)\right]\Gamma(v_L+e_L\xi)}x^{-\xi}F(1-\xi)d\xi=g_2(x)$$

where

$$(2.10)\quad g_2(x)=\Re[b_2-\mu_2,\mu_2/f_2:1/f_2]g_1(x).$$

In a similar manner, we have

$$(2.11)\quad \frac{1}{2\pi i}\int_L \frac{\prod_{j=3}^{p}\Gamma(b_j+f_j\xi)\Gamma(\mu_1+f_1\xi)\Gamma(\mu_2+f_2\xi)\prod_{j=1}^{q}\Gamma(1-a_j-e_j\xi)}{\sum_{i=1}^{r}\left[\prod_{j=p+1}^{s_i}\Gamma(1-b_{ji}-f_{ji}\xi)\prod_{j=q+1}^{t_i-1}\Gamma(a_{ji}+e_{ji}\xi)\right]\Gamma(v_L+e_L\xi)}x^{-\xi}F(1-\xi)d\xi=g_3(x),$$

$$g_3(x)=\Re[V_2-a_L,a_L/e_L:1/e_L]g_2(x)$$

and so on.

Again making use of Perseval's theorem, we obtain from (2.7), (2.9),(2.11)

respectively

$$(2.12)\quad g_1(x)=\int_0^\infty I\left[ux\left|\begin{matrix}\left\{(1-a_j,e_j)_{1,q}\right\}:\left\{(a_{ji},e_{ji})_{q+1,t_i}\right\}(v_L,e_L)\\ \left\{(\mu_1,f_1)(b_j,f_j)_{2,p}\right\}\left\{(1-b_{ji},f_{ji})_{p+1,s_i}\right\}\end{matrix}\right.\right]f(u)du\,,$$

$$(2.13)\quad g_2(x)=\int_0^\infty I\left[ux\left|\begin{matrix}\left\{(1-a_j,e_j)_{1,q}\right\}:\left\{(a_{ji},e_{ji})_{q+1,t_i}\right\}(v_L,e_L)\\ \left\{(\mu_1,f_1)(u_2,f_2)\right\}\left\{(b_j,f_j)_{3,p}\right\}\left\{(1-b_{ji},f_{ji})_{p+1,s_i}\right\}\end{matrix}\right.\right]f(u)du\,,$$

$$(2.14)\quad g_3(x)=\int_0^\infty I\left[ux\left|\begin{matrix}\left\{(1-a_j,e_j)_{1,q}\right\}:\left\{(a_{ji},e_{ji})_{q+1,t_i}\right\}(v_L,e_L)\\ \left\{(\mu_1,f_1)(u_2,f_2)\right\}\left\{(b_j,f_j)_{3,p}\right\}\left\{(1-b_{ji},f_{ji})_{p+1,s_i}\right\}\end{matrix}\right.\right]f(u)du\,,$$

where the functions $g_1(x), g_2(x)$ and $g_3(x)$ are given in (2.12), (2.13) and (2.14).

Now, we are in a position to split either of the form (2.13), (2.14), (2.15), whichever is suitable, into the pair of known integral equations. We have already indicated that the choice of the form depends on the type and number of parameters in (2.1). The criteria for the selection will be clear when we arrive at the final solution.

First we assure that any one of $f_j\,(j=1,2,...p)$ is equal to unity or reducible to unity say $f_1=1$, where $\mu_1=0$.

In this case we consider the integral equation (2.11) which may be written in the form

$$(2.15)\quad \int_0^\infty I\left[ux\left|\begin{matrix}\left\{(1-a_j,e_j)_{1,q}\right\}:\left\{(a_{ji},e_{ji})_{q+1,t_i}\right\}\\ \left\{(0,1)\right\},\left\{(b_j,f_j)_{2,p}\right\}\left\{(1-b_{ji},f_{ji})_{p+1,s_i}\right\}\end{matrix}\right.\right]f(u)du=g_1(x).$$

Now we use the following known result

$$(2.16)\quad \int_0^\infty e^{-us} I\left[\frac{u}{z}\left|\begin{matrix}\left\{(1-A_j,E_j)_{1,Q}\right\}:\left\{(A_{ji},E_{ji})_{Q+1,T_i}\right\}\\ \left\{(B_j,F_j)_{1,P}\right\},\left\{(1-B_{ji},F_{ji})_{P+1,S_i}\right\}\end{matrix}\right.\right]du$$

$$= \frac{1}{s} I \left[\frac{1}{zs} \middle| \begin{matrix} (1,1)\{(1-A_j, E_j)_{1,Q}\} : \{(A_{ji}, E_{ji})_{Q+1,T_i}\} \\ \{(B_j, F_j)_{1,P}\}, \{(1-B_{ji}, F_{ji})_{P+1,S_i}\} \end{matrix} \right].$$

The I-function converges absolutely in the ξ-plane if

(2.17) (i) $\lambda > 0, |\arg z| < \pi\lambda/2$,

(ii) $\lambda \geq 0, |\arg z| \geq \pi\lambda/2, \mathrm{Re}(\mu+1) < 0$,

$$\lambda = \sum_{j=1}^{Q} E_j + \sum_{j=1}^{P} F_j - \max_{1\leq i\leq r} \left[\sum_{j=p+1}^{Si} E_{ji} + \sum_{j=Q+1}^{Ti} F_{ji} \right],$$

$$\mu = \sum_{j=1}^{P} (B_j - F_j\xi) - \sum_{j=1}^{Q} (A_j - E_j\xi) - \max_{1\leq i\leq r} \left[\sum_{j=Q+1}^{T_i} (A_j - E_j\xi) - \sum_{j=P+1}^{S_i} (B_j - F_j\xi) + \frac{S_i}{2}, \frac{T_i}{2} \right].$$

If $\xi = 0$ then the set of above expressions (2.16) and (2.17) reduce to

$$\lambda = \sum_{j=1}^{P} E_j + \sum_{j=1}^{Q} F_j - \max_{1\leq i\leq r} \left[\sum_{j=P+1}^{S_i} E_{ji} + \sum_{j=Q+1}^{T_i} F_{ji} \right],$$

$$\mu = \sum_{j=1}^{P} (B_j) - \sum_{j=1}^{Q} (A_j) - \max_{1\leq i\leq rf} \left[\sum_{j=Q+1}^{T_j} (A_j) - \sum_{j=P+1}^{S_j} (B_j) - S_i/2, T_i/2 \right].$$

But by using the formulas

(2.18) $$I \left[x \middle| \begin{matrix} \{(1-A_j, E_j)_{1,Q}\}, \{(A_{ji}, E_{ji})_{Q+1,T_i}\} \\ \{(B_j, F_j)_{1,P}\}, \{(1-B_{ji}, F_{ji})_{P+1,S_i}\} \end{matrix} \right] = I \left[\frac{1}{x} \middle| \begin{matrix} \{(B_j, F_j)_{1,P}\}, \{(1-B_{ji}, F_{ji})_{P+1,S_i}\} \\ \{(1-A_j, E_j)_{1,Q}\}, \{(A_{ji}, E_{ji})_{Q+1,T_i}\} \end{matrix} \right]$$

and

(2.19) $$x^{\rho} I \left[x \middle| \begin{matrix} \{(1-A_j, E_j)_{1,Q}\}, \{(A_{ji}, E_{ji})_{Q+1,T_i}\} \\ \{(B_j, F_j)_{1,P}\}, \{(1-B_{ji}, F_{ji})_{P+1,S_i}\} \end{matrix} \right]$$

$$= I\left[x\left|\begin{matrix}\left\{(1-A_j-\rho E_J,E_J)_{1,Q}\right\}:\left\{(A_{Ji},+\rho E_{Ji},E_{Ji})_{Q+1,T}\right\}\\ \left\{(B_J+\rho F_J,F_J)_{1,P}\right\},\left\{(1-B_{Ji}-\rho F_{Ji},F_{Ji})_{P+1,S}\right\}\end{matrix}\right.\right],$$

we can transform (2.15) to

$$(2.20)\quad \int_0^\infty e^{-us}\left[\frac{u}{z}\left|\begin{matrix}\left\{(1-A_j,E_j)_{1,Q}\right\},\left\{(A_{ji},E_{ji})_{Q+1,T_i}\right\}\\ \left\{(B_j,F_j)_{1,P}\right\},\left\{(1-B_{ji},F_{ji})_{P+1,S_i}\right\}\end{matrix}\right.\right]du$$

$$= zI\left[zs\left|\begin{matrix}\left\{(B_j+F_j,F_j)_{1,P}\right\},\left\{(1-B_{ji}-F_{ji},F_{ji})_{P+1,S_i}\right\}\\ (0,1)\left\{(1-A_j-E_j,E_j)_{1,Q}\right\},\left\{(A_{ji}+E_{ji},E_{ji})_{Q+1,T_i}\right\}\end{matrix}\right.\right].$$

Applying equation (2.20) in (2.15), we get

$$(2.21)\quad \int_0^\infty f(u)\int_0^\infty e^{-uy}I\left[xy\left|\begin{matrix}\left\{(1-A_j,E_j)_{1Q_1}\right\},\left\{(A_{ji},E_j)_{Q_i+1,T_i}\right\}\\ \left\{(B_j,F_j)_{1,P_1}\right\},\left\{(1-B_{ji},F_{ji})_{P_1+1,S_1}\right\}\end{matrix}\right.\right]dydu$$

and $G_1(x)=\frac{1}{x}g_1(1/x)$.

Reversing the order of integration in (2.21) we arrive at the following set of integral equations.

$$(2.22)\quad \int_0^\infty e^{-ux}f(u)du = h_1(x),$$

$$(2.23)\quad \int_0^\infty I\left[ux\left|\begin{matrix}\left\{(1-A_j,E_j)_{1,Q_1}\right\}:\left\{(A_{ji},E_{ji})_{Q_1+1,T_{ij}}\right\}\\ \left\{(B_j,F_j)_{1,P_1}\right\}:\left\{(1-B_{ji},E_{ji})_{P_1+1,S_{ij}}\right\}\end{matrix}\right.\right]h_1(u)du = G_1(x)\ .$$

Clearly, is Laplace transform of $f(u)$. Therefore the inversion formula yields

$$(2.24)\quad f(u)=\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}e^{uz}h_1(z)dz,$$

where $\sigma_0 = \mathrm{Re}(z)$

Hence, if the integral equation (2.22) is assumed to have known inversion, then $h_1(x)$ will be obtained in terms of prescribed function $G_1(x)$. Consequently the uknown function $f(x)$ will be determined with the help of (2.24). Actually the selection of the form (2.11) implies that the resulting integral equation (2.22) is standard one. Accordingly we have the following

Theorem. If $f(x)$ and $h_1(x)$ are solutions of the integral equations

$$(2.25)\quad g(x) = \int_0^\infty I\left[ux\left|\begin{matrix}\{(1-a_j,e_j)_{1,q}\}:\{(a_{ji},e_{ji})_{q+1,t_i}\}\\ \{(b_1,1)\},\{(b_j,f)_{2,p}\}\{1-b_{ji},f_{ji}\}_{p+1,s_1}\end{matrix}\right.\right]f(u)du = g_1(x)$$

and

$$(2.26)\quad G_l(x) = \int_0^\infty I\left[ux\left|\begin{matrix}\{(1-A_j,E_j)_{1,Q_1}\}:\{(A_{ji},E_{ji})_{Q_1+1,T_{il}}\}\\ \{(B_1,F_j)_{1,P_l}\},\{1-B_{ji},F_{ji}\}_{p_l+1,s_{il}}\end{matrix}\right.\right]h_1(u)du$$

respectively, then

$$(2.27)\quad f(u) = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} e^{zx}h_l(z)dz,$$

where

$$(2.28)\quad G_l(x) = \frac{1}{x}R[b_1,0;1]g(1/x),$$

under the existence conditions of I-function.

Special Cases. Now, we proceed to the transform of (2.14) in the desired form. We assume that any two of $f_j\,(j=1,2,\ldots,p)$ are equal to 1/2 or reducible to 1/2. For convenience we take $f_1 = f_2 = 1/2$. Since the constants μ_1 and μ_2 are arbitrarily chosen, we put

$$\mu_1 = 1/4 + \upsilon/2, \mu_2 = 1/4 - \upsilon/2.$$

Hence the relation (2.13) takes the form

$$(2.29) \int_0^\infty I\left[ux\left|\begin{matrix}\{(1-a_j,e_j)_{1,q}\}:\{(a_{ji},e_{ji})\}_{q+1,t_i} \\ (1/4+v/2,1/2),(1/4-v/2,1/2)\{(b_j,f_j)_{2,p}\}\{1-b_{ji},f_{ji}\}_{p+1,s_i}\end{matrix}\right.\right]f(u)du = g_2(x)$$

Using Bessel's formula

$$(2.30) \int_0^\infty K_v(\beta t)t^{\mu-1}dt = 2^{\mu-2}\times\beta^{-\mu}\Gamma(\mu/2+v/2)\times\Gamma(\mu/2-v/2),$$

we get the following integral

$$(2.31)\ \frac{1}{2\sqrt{2}s}I\left[\frac{1}{zs}\left|\begin{matrix}(3/4+v/2,1/2),(3/4-v/2,1/2),\{(1-A_j,E_j)_{Q_1}\},\{(A_{ji},E_{ji})_{Q+1,T_i}\} \\ \{(B_j,F_j)_{1,P}\},\{(1-B_{ji},F_{ji})_{P+1,S_i}\}\end{matrix}\right.\right],$$

provided $\mathrm{Re}(s)>0, \mathrm{Re}(3/2\pm v+\min B_j/F_j)>0 (j=1,2,\ldots P)$.

Making use of this integral (2.28), we have

$$(2.32) \int_0^\infty f(u)\int_0^\infty (2uy)^{1/2}K_v(2uy)I\left[xy\left|\begin{matrix}\{(1-A_j,E_j)_{Q_2}\},\{(A_{ji},E_{ji})_{Q_2+1,T_{i2}}\} \\ \{(B_j,F_j)_{1,P_2}\},\{(1-B_{ji},F_{ji})_{P_2+1,S_{i2}}\}\end{matrix}\right.\right]dydu = G_2(x),$$

where $G_2(x) = \dfrac{1}{2\sqrt{2}x}g_2(1/x)$.

Finally the reversal of integration in (2.33) gives the following pair of integral equations

$$(2.33)\ \int_0^\infty (2ux)^{1/2}K_v(2ux)f(u)du = h_2(x),$$

$$(2.34)\ \int_0^\infty I\left[ux\left|\begin{matrix}\{(1-A_j,E_j)_{1,Q_2}\}:\{(A_{ji},E_{ji})_{Q_2+1,T_{i2}}\} \\ \{(B_j,F_j)_{1,P_2}\},\{(1-B_{ji},F_{ji})_{P_2+1,S_{i2}}\}\end{matrix}\right.\right]h_2(u)du = G_2(x).$$

Obviously $h_2(x)$ is Meijer's Bessel function transform of $f(x)$. Therefore applying the inversion formula of Laplace transform, we get

$$(2.35)\ \ f(x) = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}(2zx)^{1/2}I_v(2zx)h_2(z)dz,$$

where $\sigma_0 = \text{Re}(z)$.

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PATTERN AND GROWTH OF ANIMAL POPULATION WITH THREE AGE GROUPS

By

Vimlesh Chaturvedi

Laxmi Narayan Institute of Technology, Gwalior, Madhya Pradesh, India

E-mail :vkc1012@gmail.com

V.P. Saxena

Sagar Institute of Research Technology of Science, Bhopal-462041, Madhya Pradesh, India

E-mail :vinodpsaxena@gmail.com

and

Padam Sharma

Sagar Institute of Research Technology of Science, Bhopal-462041, Madhya Pradesh, India

E-mail :ppadam83@gmail.com

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ABSTRACT

In this paper we consider certain elementary models of finite but new population models of non-interacting species. This population is divided in three age groups, namely juvenile, reproductive and non-reproductive. It is assumed that the three classes maintain a fixed ratio throughout the period under consideration. The length of each regular subinterval is called a generation and depends on the particular animal or insect. The formulation is in terms of a system of linear finite difference equations incorporating birth rate, death rate and migration. The solutions are obtained in terms of finite polynomials having variable dependent on the important parameters. Graphs have been plotted for various sets of migration rate for different generations.

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1. Introduction. Population dynamics is an important area of biomathematics. The population studies are associated with many processes in life sciences, such as natural growth and culture of microorganisms, growth and decay of viruses, growth, decay and migration in aquatic and wild life and demography. These studies differ from one population species to other depending on the type of the growth and environment involved. Some populations are isolated and grow independently with each unit dividing itself or grow out of heterogeneous

pair contributing to reproduction. While in other cases two or more species of population live together and may be dependent or independent on each other. In all cases the environment plays a key role along with other factors such as availability of nutrients, presence of toxicants and physic-chemical condition. Thus, population dynamics is a subject with many applications. It also plays role in other studies like epidemiology, pollution analysis and ecology. In particular, it plays a major role in the management and manipulation of protected wild life.

Any biological population has the following important factors:

* Growth/Reproduction
* Decay/Death

The Mathematical behaviour has the following types with respect to time

* Discrete
* Continuous
* Piece-wise continuous
* Transitional

The mathematical modeling of each may differ with others in seveal respects. This difference may be due to various factors like natality, mortality, fertility, community behavior, the impact of environment and genetic effect.

Populations are groups of individuals belonging to the same species that live in the same region at the same time. Populations, like individual organisms, have unique attributes such as:

* growth rate
* age structure
* sex ratio
* mortality rate

If a population A of the same species has finite number of individuals (possibility of both the sexes and all the normal age groups) is called a finite population. In this case A is discrete and is governed by difference equation Gotteli [1].

$$A_n = A_{n-1} + f(A_{n-1}, \lambda),$$

where A_n denotes population after n generations with A_0 as its initial value is an appropriate function depending on the type of species and environment, λ is control parameter. If f is linear, say $f = \alpha A_{n-1} + \lambda$, then

$$A = \alpha^n A_0 + \lambda\left(1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1}\right) = \alpha^n A_0 + \lambda \sum_{i=0}^{n-1} \alpha^i .$$

In a prescribed domain of habitat the size of population determines the

approach for quantitative modeling. In many cases we have to consider a given population size as discrete number and its growth as a step function of time. In such situations the population size is relatively small and the growth rate is generally fixed during a given time interval. This interval we call a step or a generation. For all practical purposes we take this interval as a time unit. The population under consideration is self-contained in healthy environment and has enough amount of nutrition available. There are few population studies in which applications of special functions are classical growth of finite populations Saxena [3]. In This paper we make an attempt to express structured finite population in terms of polynomials of two variables. These polynomials are close to Hermite polynomials of two variables Khan, M.A. et al. [2], Xu [5].

Formulation of the Problem. We assume that A_n denotes the population size after n^{th} steps or generations with initial population A_0. Medium and large animal species with longer generation of life the population has to be divided in at least three groups; we can take Vaishya [4].

$$A_n = A_n^{(1)} + A_n^{(2)} + A_n^{(3)} \quad \text{...(1)}$$

where

$A_n^{(1)}$ = Population of infants and juveniles (pre fertile),

$A_n^{(2)}$ = Adult population in the fertile age group,

$A_n^{(3)}$ = Population of aged non-fertile.

The change in population of first age group will take due to births, deaths and migration. Therefore. the change in the population $A_n^{(1)}$ is given by the equation

$$\Delta A_n^{(1)} = A_n^{(1)} - A_{n-1}^{(1)}.$$

The governing equation can be written as

$$\Delta A_n^{(1)} = B_1 A_{n-1}^{(2)} - D_1 A_{n-1}^{(1)} - T_1 A_{n-1}^{(1)} - q_1 Q A_{n-1}^{(1)} + M_1.$$

Accordingly, we get the following difference equation

$$A_n^{(1)} = (1 - D_1 - T_1 - q_1 Q) A_{n-1}^{(1)} + B_1 A_{n-1}^{(2)} + M_1. \quad \text{...(2)}$$

Similarly, we have

$$\Delta A_n^{(2)} = A_n^{(2)} - A_{n-1}^{(2)}.$$

The governing difference equation is

$$A_n^{(2)} = (1 - D_2 - T_2 - q_2 Q) A_{n-1}^{(2)} + T_1 A_{n-1}^{(1)} + M_2. \quad \text{...(3)}$$

Similarly, we can write

$$A_n^{(3)} = (1 - D_3 - q_3Q)A_{n-1}^{(3)} + T_2A_{n-1}^{(2)} + M_3 \qquad ...(4)$$

where B_i, D_i, T_i, M_i and q_i are respectively birth rate, death rate, transition rate, migration rate and predation rate in i^{th} generation. Here we have assumed that $B_2 = B_3 = 0$ as there is no contribution of birth in second and third age groups. Also the entire population is exposed to the predators having by and large a slow growing number of predators represented by Q.

Solution of the Problem. Let

$$1 - D_1 - T_1 - q_1Q = x.$$

Then, equation (2) becomes

$$A_n^{(1)} = xA_{n-1}^{(1)} + B_1A_{n-1}^{(2)} + M_1. \qquad ...(5)$$

Further, let $1 - D_2 - T_2 - q_2Q = y$.

Then, equation (3) becomes

$$A_n^{(2)} = yA_{n-1}^{(2)} + T_1A_{n-1}^{(1)} + M_2, \qquad ...(6)$$

and for $1 - D_3 - q_3Q = z$, equation (4) becomes

$$A_n^{(3)} = zA_{n-1}^{(3)} + T_2A_{n-1}^{(2)} + M_3. \qquad ...(7)$$

We find the solution of the equations (5), (6) and (7) by induction method, i.e.

$$A_1^{(1)} = xA_0^{(1)} + B_1A_0^{(2)} + M_1, \qquad ...(8)$$

$$A_1^{(2)} = yA_0^{(2)} + T_1A_0^{(1)} + M_2, \qquad ...(9)$$

$$A_2^{(1)} = (x^2 + B_1T_1)A_0^{(1)} + B_1(x+y)A_0^{(2)} + (1+x)M_1 + B_1M_2, \qquad ...(10)$$

$$A_2^{(2)} = yA_1^{(2)} + T_1A_1^{(1)} + M_2.$$

Putting the values of $A_1^{(2)}$ and $A_1^{(1)}$, we get

$$A_2^{(2)} = (y^2 + B_1T_1)A_0^{(2)} + T_1(x+y)A_0^{(1)} + (1+y)M_2 + T_1M_1, \qquad ...(11)$$

$$A_3^{(1)} = xA_2^{(1)} + B_1A_2^{(2)} + M_1.$$

Let us define

$$p_n(x) = 1 + x + x^2 + ... + x^n$$

and

$$q_n(x) = y + y^2 + ... + y^n.$$

Then, putting the values of $A_2^{(1)}$ and $A_2^{(2)}$, we get

$$A_3^{(1)} = x^3 A_0^{(1)} + B_1 T_1 (2x + y) A_0^{(1)} + B_1 A_0^{(2)} \sum_{r=0}^{2} x^{2-r} y^r + B_1^2 T_1 A_0^{(2)} + M_1 p_2(x)$$

$$+ B_1 M_2 \left(p_1(x) + q_1(y)\right) + T_1 B_1 M_1 . \qquad ...(12)$$

Similarly, in the equation

$$A_3^{(2)} = y A_2^{(2)} + T_1 A_2^{(1)} + M_2 ,$$

putting the values of $A_2^{(1)}$ and $A_2^{(2)}$, we get

$$A_3^{(2)} = y^3 A_0^{(2)} + B_1 T_1 (2y + x) A_0^{(2)} + T_1 A_0^{(1)} \sum_{r=1}^{3} x^{3-r} y^{r-1} + B_1 T_1^2 A_0^{(1)} + M_2 \left(1 + q_2(y)\right)$$

$$+ \left(p_1(x) + q_1(y)\right) T_1 M_1 + B_1 T_1 M_2 . \qquad ...(13)$$

Likewise

$$A_4^{(1)} = x A_3^{(1)} + B_1 A_3^{(2)} + M_1 .$$

Putting the values of $A_3^{(1)}$ and $A_3^{(2)}$, we get

$$A_4^{(1)} = x^4 A_0^{(1)} + B_1 T_1 A_0^{(1)} \sum_{r=0}^{2} (3-r) x^{2-r} y^r + B_1^2 T_1^2 A_0^{(1)} + B_1 A_0^{(2)} \sum_{r=1}^{3} x^{4-r} y^{r-1} + B_1^2 T_1 (2x + 2y) A_0^{(2)}$$

$$+ M_1 p_3(x) + B_1 T_1 M_1 (1 + 2x + y) + B_1 M_2 \left(p_2(x) + xy + q_2(y)\right) + B_1^2 T_1 M_2 . \qquad ...(14)$$

Again,

$$A_4^{(2)} = y A_3^{(2)} + T_1 A_3^{(1)} + M_2 .$$

Putting the values of $A_3^{(1)}$ and $A_3^{(2)}$, we get

$$A_4^{(2)} = y^4 A_0^{(2)} + B_1 T_1 A_0^{(2)} \sum_{r=1}^{3} (4-r) x^{r-1} y^{3-r} + B_1^2 T_1^2 A_0^{(2)} + T_1 A_0^{(1)} \sum_{r=1}^{4} x^{4-r} y^{r-1} + 2 B_1 T_1^2 (x + y) A_0^{(1)}$$

$$+ M_2 \left[1 + q_3(y)\right] + B_1 T_1 M_2 \left(p_1(x) + 2 q_1(y)\right) + T_1 M_1 \left(p_2(x) + xy + q_2(y)\right) + B_1 T_1^2 M_1 \qquad ...(15)$$

From

$$A_5^{(1)} = x A_4^{(1)} + B_1 A_4^{(2)} + M_1 .$$

Putting the values of A_4^1 and $A_4^{(2)}$, we get

$$A_5^{(1)} = x^5 A_0^{(1)} + B_1 T_1 A_0^{(1)} \sum_{r=1}^{4} (5-r) x^{4-r} y^{r-1} + B_1^2 T_1^2 (3x + 2y) A_0^{(1)} + B_1 A_0^{(2)} \sum_{r=1}^{5} x^{5-r} y^{r-1}$$

$$+B_1^2T_1\left(3x^2+4xy+3y^2\right)A_0^{(2)}+B_1^3T_1^2A_0^{(2)}+M_1p_4(x)+B_1T_1\left(\sum_{r=1}^{2}rx^r+p_2(x)+2xy+q_2(y)\right)M_1$$

$$+B_1\left(p_3(x)+xy+\sum_{r=1}^{2}x^{3-r}y^r+q_3(y)\right)M_2+B_1^2T_1\left(1+2\sum_{r=1}^{2}x^{2-r}y^{r-1}\right)M_2+B_1^2T_1^2M_1\,. \quad \ldots(16)$$

From

$$A_5^{(2)}=yA_4^{(2)}+T_1A_4^{(1)}+M_2\,,$$

putting the values of $A_4^{(1)}$ and $A_4^{(2)}$, we get

$$A_5^{(2)}=y^5A_0^{(2)}+B_1T_1A_0^{(2)}\sum_{r=0}^{3}(4-r)y^{3-r}x^r+B_1^2T_1^2(2x+3y)A_0^{(2)}+T_1A_0^{(1)}\sum_{r=1}^{5}x^{r-1}y^{5-r}$$

$$+B_1T_1^2\left(3x^2+4xy+3y^2\right)A_0^{(1)}+B_1^2T_1^3A_0^{(1)}+M_2\left[1+q_4(y)\right]$$

$$+B_1T_1\left(\sum_{r=1}^{2}ry^r+1+q_2(y)+2xy+p_2(x)-1\right)M_2$$

$$+T_1\left(p_3(x)+xy+\sum_{r=1}^{2}x^{3-r}y^r+q_3(y)\right)M_1+B_1T_1^2\left(1+2\sum_{r=1}^{2}x^{2-r}y^{r-1}\right)M_1+B_1^2T_1^2M_2\,. \quad \ldots(17)$$

From

$$A_6^{(1)}=xA_5^{(1)}+B_1A_5^{(2)}+M_1\,,$$

putting the values of $A_5^{(1)}$ and $A_5^{(2)}$, we get

$$A_6^{(1)}=x^6A_0^{(1)}+B_1T_1A_0^{(1)}\sum_{r=0}^{4}(5-r)x^{4-r}y^r+B_1^2T_1^2\left(6x^2+6xy+3y^2\right)A_0^{(1)}$$

$$+B_1^3T_1^3A_0^{(1)}+B_1A_0^{(2)}\sum_{r=0}^{5}x^{5-r}y^r+B_1^2T_1\left(4\sum_{r=0}^{3}x^{3-r}y^r+2\sum_{r=1}^{2}x^{3-r}y^r\right)A_0^{(2)}$$

$$+B_1^3T_1^2(3x+3y)A_0^{(2)}+M_1p_5(x)+B_1^2T_1^2(1+3x+2y)M_1$$

$$+B_1T_1\left(\sum_{r=1}^{3}rx^r+p_3(x)+\sum_{r=0}^{1}(3-r)x^{2-r}y^{r+1}+2xy+\sum_{r=2}^{3}y^r\right)M_1$$

$$+B_1\left(p_4(x)+\sum_{r=0}^{1}x^{2-r}y^{r+1}+\sum_{r=0}^{2}x^{3-r}y^{r+1}+xy+q_4(y)\right)M_2+B_1^3T_1^2M_2$$

$$+B_1^2T_1\left(\sum_{r=1}^{2}rx^r+p_2(x)+4xy+\sum_{r=1}^{2}ry^r+q_2(y)\right)M_2\,. \quad \ldots(18)$$

From

$$A_6^{(2)} = yA_5^{(2)} + T_1 A_5^{(1)} + M_2 ,$$

putting the values of $A_5^{(1)}$ and $A_5^{(2)}$, we get

$$A_6^{(2)} = y^6 A_0^{(2)} + B_1 T_1 A_0^{(2)} \sum_{r=0}^{4} (5-r) y^{4-r} x^r + 3B_1^2 T_1^2 A_0^{(2)} \left[\sum_{r=0}^{2} x^r y^{2-r} + \sum_{r=0}^{1} x^r y^{2-r} \right]$$

$$+ B_1^3 T_1^3 A_0^{(2)} + T_1 A_0^{(1)} \left[\sum_{r=0}^{5} x^r y^{5-r} + 2x^2 y^3 \right] + 3B_1^2 T_1^3 A_0^{(1)} (x+y)$$

$$+ B_1 T_1^2 A_0^{(1)} \left[4\sum_{r=0}^{3} x^{3-r} y^r + 2\sum_{r=1}^{2} x^{3-r} y^r \right] + M_2 \left[1 + q_5(y) \right]$$

$$+ B_1 T_1 M_2 \left[\sum_{r=0}^{3} (r+1) y^r + 2y\{p_2(x) - 1\} + 3xy^2 + p_3(x) - 1 \right]$$

$$+ T_1 M_1 \left[p_4(x) + x^3 y + x^2 q_2(y) + x q_3(y) + q_4(y) \right] + B_1^2 T_1^2 M_2 (1 + 2x + 3y)$$

$$+ B_1 T_1^2 M_1 \left[\sum_{r=0}^{2} (r+1) x^r + 4xy + 2q_2(y) + y^2 \right] + B_1^2 T_1^3 M_1 . \qquad \text{...(19)}$$

From

$$A_7^{(1)} = xA_6^{(1)} + B_1 A_6^{(2)} + M_1 ,$$

putting the values of $A_6^{(1)}$ and $A_6^{(2)}$, we get

$$A_7^{(1)} = x^7 A_0^{(1)} + B_1 T_1 A_0^{(1)} \left[\sum_{r=0}^{4} (5-r) x^{5-r} y^r + \sum_{r=0}^{5} x^r y^{5-r} + 2x^2 y^3 \right]$$

$$+ B_1^2 T_1^2 A_0^{(1)} \left[4\sum_{r=0}^{3} x^{3-r} y^r + 5\sum_{r=0}^{2} x^{3-r} y^r + x^3 + 3x^2 y \right] + B_1^3 T_1^3 A_0^{(1)} (4x + 3y)$$

$$+ B_1 A_0^{(2)} \sum_{r=0}^{6} x^{6-r} y^r + B_1^2 T_1 A_0^{(2)} \left[5\sum_{r=0}^{4} x^{4-r} y^r + 3\sum_{r=1}^{3} x^{4-r} y^r + x^2 y^2 \right]$$

$$+ B_1^3 T_1^2 A_0^{(2)} \left[6\sum_{r=0}^{2} x^{2-r} y^r + 3xy \right] + M_1 p_6(x)$$

$$+ B_1 T_1 M_1 \left[\sum_{r=0}^{4} (r+1) x^r + 2\sum_{r=1}^{3} x^{4-r} y^r + \sum_{r=1}^{2} x^{4-r} y^r + y\{p_3(x) - 1\} + q_4(y) + 2\sum_{r=1}^{2} x^{3-r} y^r \right]$$

$$+ B_1^2 T_1^2 M_1 \left[\sum_{r=0}^{2} (r+1) y^r + 3x + 6x^2 + 6xy \right]$$

$$+B_1M_2\left[p_5(x)+\{p_3(x)-1\}q_2(y)+y^3\{p_2(x)-1\}+xy(x^3+y^3)+q_5(y)\right]$$

$$+B_1^2T_1M_2\left[\sum_{r=0}^{3}(r+1)x^r+6x^2y+4xy+6xy^2+\sum_{r=1}^{3}(r+1)y^r\right]$$

$$+B_1^3T_1^2M_2[1+3x+3y]+B_1^4T_1^3A_0^{(2)}+B_1^3T_1^3M_1. \qquad ...(20)$$

From

$$A_7^{(2)}=yA_6^{(2)}+T_1A_6^{(1)}+M_2,$$

putting the values of $A_6^{(1)}$ and $A_6^{(2)}$, we get

$$A_7^{(2)}=y^7A_0^{(2)}+B_1T_1A_0^{(2)}\sum_{r=0}^{5}(6-r)x^ry^{5-r}+B_1^2T_1^2A_0^{(2)}\left[4\sum_{r=0}^{3}x^ry^{3-r}+5\sum_{r=0}^{2}x^ry^{3-r}+\sum_{r=1}^{2}(2^r-1)x^{r-1}y^{4-r}\right]$$

$$+B_1^3T_1^3A_0^{(2)}(3x+4y)+T_1A_0^{(1)}\left[\sum_{r=0}^{6}x^ry^{6-r}+2x^2y\right]+B_1T_1^2A_0^{(1)}\left[5\sum_{r=0}^{4}x^{4-r}y^r+3\sum_{r=1}^{3}x^{4-r}y^r+x^2y^2\right]$$

$$+B_1^2T_1^3A_0^{(1)}\left[6\sum_{r=0}^{2}x^{2-r}y^r+3xy\right]+M_2\left[1+q_6(y)\right]$$

$$+B_1T_1M_2\left[\sum_{r=1}^{4}(r+1)y^r+3xy^2p_1(x)+4xy^3+2xyp_2(x)+p_4(x)\right]$$

$$+T_1M_1\left[q_5(y)+xyp_3(x)+xy^2p_2(x)+y^3\{p_2(x)-1\}+xy^2p_5(x)\right]$$

$$+B_1^2T_1^2M_2\left[\sum_{r=1}^{3}rx^{r-1}+3y(1+2x+2y)\right]+B_1T_1^2M_1\left[q_3(y)+4xy+6xy(x+y)+2y^2+\sum_{r=1}^{4}rx^{r-1}\right]$$

$$+B_1^2T_1^3M_1(1+3x+3y)+B_1^3T_1^4A_0^{(1)}+B_1^3T_1^3M_2. \qquad ...(21)$$

Now, from equation (7) we get

$$A_n^{(3)}=zA_{n-1}^{(3)}+T_2A_{n-1}^{(2)}+M_3.$$

Therefore, $A_1^{(3)}=zA_0^{(3)}+T_2A_0^{(2)}+M_3$. ...(22)

From

$$A_2^{(3)}=z^2A_0^{(3)}+T_2(z+y)A_0^{(2)}+T_1T_2A_0^{(1)}+(1+z)M_3+T_2M_2 \qquad ...(23)$$

$$A_3^{(3)}=zA_2^{(3)}+T_2A_2^{(2)}+M_3$$

Let,

$$r_n(z)=1+z+z^2+...+z^n.$$

Then, putting the values of $A_2^{(3)}$ and $A_2^{(2)}$, we get

$$A_3^{(3)} = z^3 A_0^{(3)} + T_2 A_0^{(2)} \sum_{r=0}^{2} z^{2-r} y^r + B_1 T_1 T_2 A_0^{(2)} + T_1 T_2 (x+y+z) A_0^{(1)} + r_2(z) M_3$$

$$+ T_2 M_2 \left[r_1(z) + y \right] + T_1 T_2 M_1 . \quad ...(24)$$

From

$$A_4^{(3)} = z A_3^{(3)} + T_2 A_3^{(2)} + M_3 ,$$

putting the values of $A_3^{(3)}$ and $A_3^{(2)}$, we get

$$A_4^{(3)} = z^4 A_0^{(3)} + T_2 A_0^{(2)} \sum_{r=0}^{3} z^{3-r} y^r + B_1 T_1 T_2 (x + 2y + z) A_0^{(2)}$$

$$+ T_1 T_2 \left(\sum_{r=0}^{2} z^{2-r} x^r + \sum_{r=0}^{1} y^{2-r} z^r + xy \right) A_0^{(1)} + B_1 T_1^2 T_2 A_0^{(1)} + M_3 r_3(z)$$

$$+ T_2 \left(q_2(y) + r_2(z) + yz \right) M_2 + T_1 T_2 \left(p_1(x) + y + z \right) M_1 + B_1 T_1 T_2 M_2 . \quad ...(25)$$

From

$$A_5^{(3)} = z A_4^{(3)} + T_2 A_4^{(2)} + M_3 ,$$

putting the values of $A_4^{(3)}$ and $A_4^{(2)}$, we get

$$A_5^{(3)} = z^5 A_0^{(3)} + T_2 A_0^{(2)} \sum_{r=0}^{4} z^{4-r} y^r + B_1 T_1 T_2 A_0^{(2)} \left[zx + (y+z)^2 + y^2 + (x+y)^2 \right]$$

$$+ B_1^2 T_1^2 T_2 A_0^{(2)} + T_1 T_2 A_0^{(1)} \left[\sum_{r=0}^{2} z^{3-r} x^r + yz(x+y+z) + \sum_{r=0}^{3} x^{3-r} y^r \right]$$

$$+ B_1 T_1^2 T_2 (z + 2x + 2y) A_0^{(1)} + M_3 r_4(z) + T_2 M_2 \left[q_3(y) + z q_2(y) + r_3(z) + yz^2 \right]$$

$$+ T_1 T_2 \left(p_2(x) + xy + xz + yz + r_2(z) - 1 + q_2(y) \right) M_1 + B_1 T_1 T_2 \left(r_1(z) + x + 2y \right) M_2$$

$$+ B_1 T_1^2 T_2 M_1 \quad ...(26)$$

From

$$A_6^{(3)} = z A_5^{(3)} + T_2 A_5^{(2)} + M_3 ,$$

putting the values of $A_5^{(3)}$ and $A_5^{(2)}$, we get

$$A_6^{(3)} = z^6 A_0^{(3)} + T_2 A_0^{(2)} \sum_{r=0}^{5} z^{5-r} y^r$$

$$+B_1T_1T_2A_0^{(2)}\left[\sum_{r=1}^{3} z^{3-r}x^r + \sum_{r=0}^{2} z^{3-r}y^r + \sum_{r=1}^{3} z^{3-r}y^r + y^2z + 2xyz + 3y^3 + 3y^2x + 2yx^2\right]$$

$$+B_1^2T_1^2T_2A_0^{(2)} + T_1^1T_2A_0^{(1)}\left[\sum_{r=0}^{4} z^{4-r}x^r + \sum_{r=1}^{4} z^{4-r}y^r + \sum_{r=1}^{3} x^r y^{4-r} + xyz(x+y+z)\right]$$

$$+B_1T_1^2T_2A_0^{(1)}\left[\sum_{r=0}^{2} z^{2-r}x^r + xz + x^2 + (x+y)^2 + 2y(x+y+z)\right]$$

$$+B_1^2T_1^3T_2A_0^{(1)} + M_3r_5(z) + T_2M_2\left[q_4(y) + r_4(z) + z\{1+q_3(y)\} + z^2q_2(y) + yz^3\right]$$

$$+T_1T_2M_1\left[p_3(x) + q_3(y) + r_3(z) - 1 + x(y+z)p_1(x) + yzr_1(z) + x\sum_{r=0}^{2} z^{2-r}y^r + y^2z\right]$$

$$+B_1T_1T_2M_2\left[r_2(z) + p_2(x) - 1 + (2y+z)p_1(x) + 2y(z+3y)\right]$$

$$+B_1T_1^2T_2M_1(1+2x+2y+z) + B_1^2T_1^2T_2M_2 . \qquad \text{...(27)}$$

Finally, from

$$A_7^{(3)} = zA_6^{(3)} + T_2A_6^{(2)} + M_3,$$

putting the values of $A_6^{(3)}$ and $A_6^{(2)}$, we get

$$A_7^{(3)} = z^7 A_0^{(3)} + T_2 A_0^{(2)} \sum_{r=0}^{6} z^{6-r} y^r$$

$$+B_1T_1T_2A_0^{(2)}\left[\sum_{r=0}^{4} z^{4-r}x^r + \sum_{r=0}^{3}(5-r)y^{4-r}z^r + xyz(2z+3y+2x) + \sum_{r=1}^{3}(5-r)y^{4-r}x^r\right]$$

$$+B_1^2T_1^2T_2A_0^{(2)}\left[\sum_{r=0}^{2} z^{2-r}y^r + x(2z+3y) + 2\sum_{r=0}^{1} y^{2-r}z^r + 3\sum_{r=0}^{2} x^r y^{2-r}\right]$$

$$+T_1T_2A_0^{(1)}\left[\sum_{r=0}^{5} z^{5-r}x^r + \sum_{r=1}^{5} z^{5-r}y^r + x\sum_{r=0}^{3} y^{4-r}z^r + x^2\sum_{r=0}^{2} y^{3-r}z^r + \sum_{r=1}^{3} x^{5-r}y^r + x^2y(y^2+xz)\right]$$

$$+B_1T_1^2T_2A_0^{(1)}\left[\sum_{r=0}^{3} z^{3-r}x^r + \sum_{r=1}^{3} z^{3-r}y^r + \sum_{r=1}^{3} rz^{3-r}x^r + \sum_{r=1}^{3} rz^{3-r}y^r + 4xyz + 6\sum_{r=1}^{2} x^{3-r}y^r\right] + M_3r_6(z)$$

$$+B_1^2T_1^3T_2A_0^{(1)}(z+3x+3y)$$

$$+T_2M_2\left[r_2(z) - 1 + zq_4(y) + r_5(z) + q_5(y) + z^2q_2(y) + z^3q_2(y) + yz^2(y^2+z^2)\right]$$

$$+T_1T_2M_1\Big[zp_3(x)+z^2p_2(x)+z^3p_1(x)+p_4(x)+q_4(y)+xq_3(y)+\sum_{r=0}^{3}z^{4-r}y^r$$

$$+\left(z+x^2\right)q_2(y)+xyzp_1(x)+yz^2p_1(x)+xy^2z\Big]+B_1T_1T_2M_2\Big[r_3(z)+z^2p_1(x)+p_3(x)$$

$$-1+xzp_1(x)+2yr_2(z)+3y^2r_1(z)+2xyr_1(z)+\sum_{r=1}^{3}(r+1)y^rx^{3-r}\Big]+B_1T_1^2T_2M_1$$

$$\Big[r_2(z)+2(x+y)r_1(z)+3\sum_{r=0}^{2}y^{2-r}x^r+xy\Big]+B_1^2T_1^2T_2M_2\big[r_1(z)+2x+3y\big]$$

$$+B_1^3T_1^3T_2A_0^{(2)}+B_1^2T_1^3T_2M_1\,. \qquad \text{...(28)}$$

Numerical Examples. Numerical calculations done with the help of Mat lab 7.5 (2007) programming and also all the graphs drawn with same.

CaseI : Example1. If $q_1=0.001, q_2=0, D_1=20/100, D_2=15/100,\quad T_1=T_2=T=1/100, B_1=25/100, A_0^{(1)}=1000, A_0^{(2)}=2000,\quad M_1=M_2=M=10,60,110,160,210,260,310, Q=300,400,500,600$

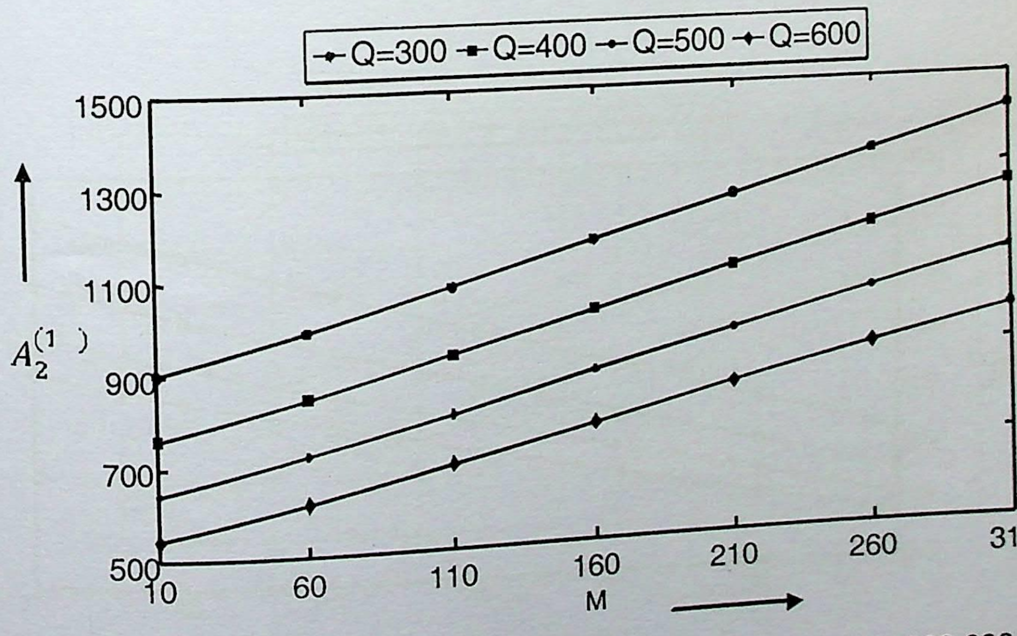


Figure 1: Graph between M and $A_2^{(1)}$ for Q= 300,400,500,600

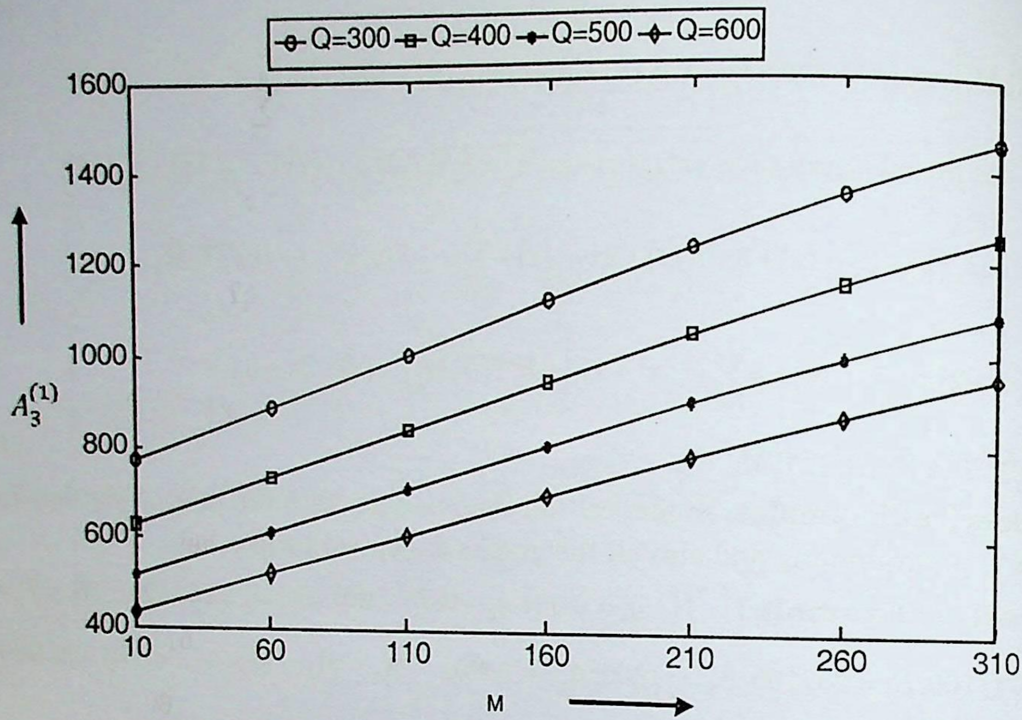


Figure-2: Graph between M and $A_3^{(1)}$ for Q=300,400,500,600

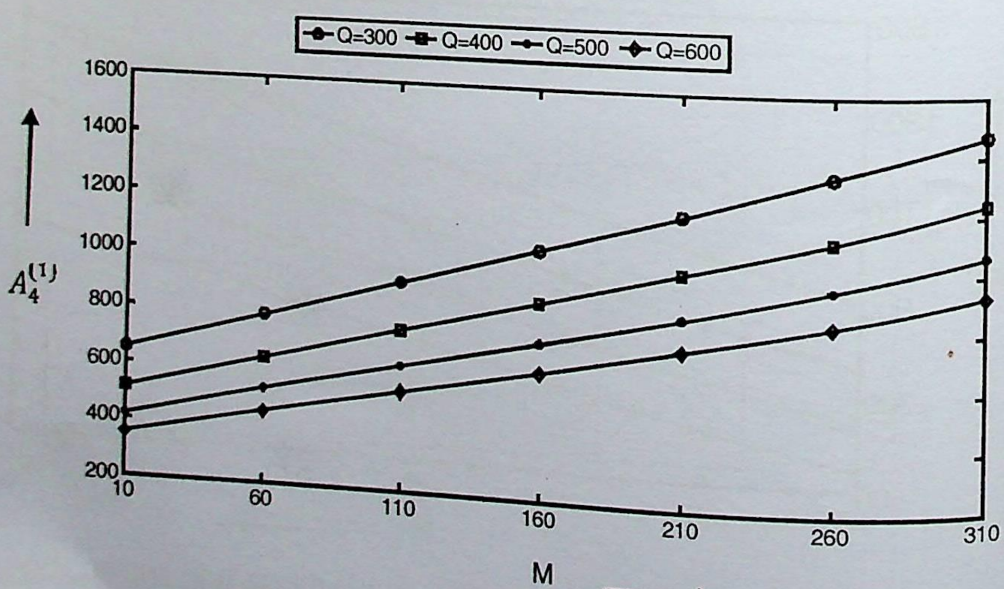


Figure-3: Graph between M and $A_4^{(1)}$ for Q=300, Q=400, Q=500, Q=600

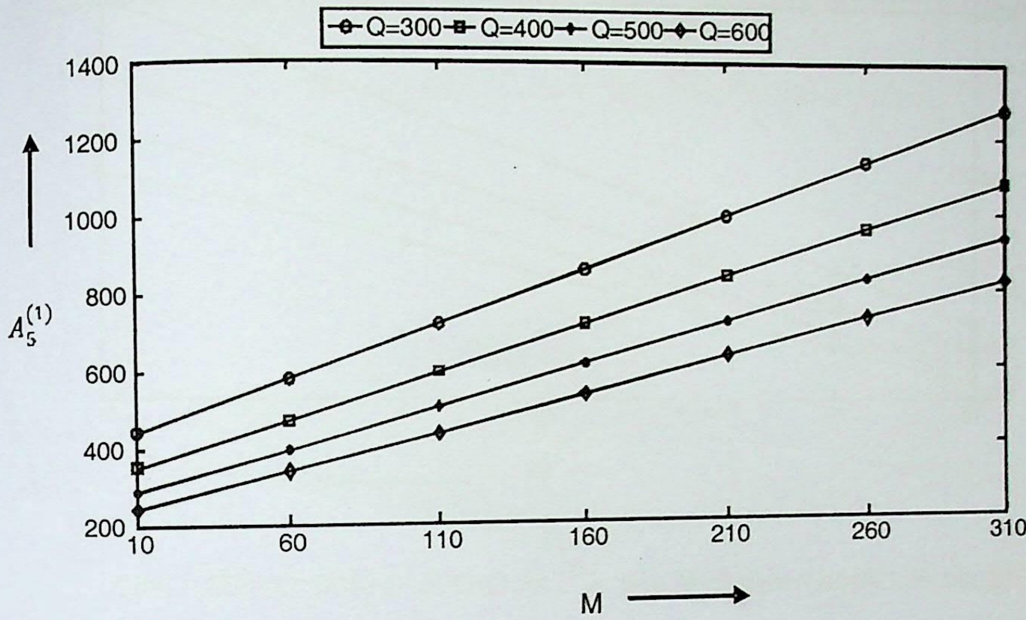


Figure-4: Graph between M and $A_5^{(1)}$ for Q=300, Q=400, Q=500, Q=600

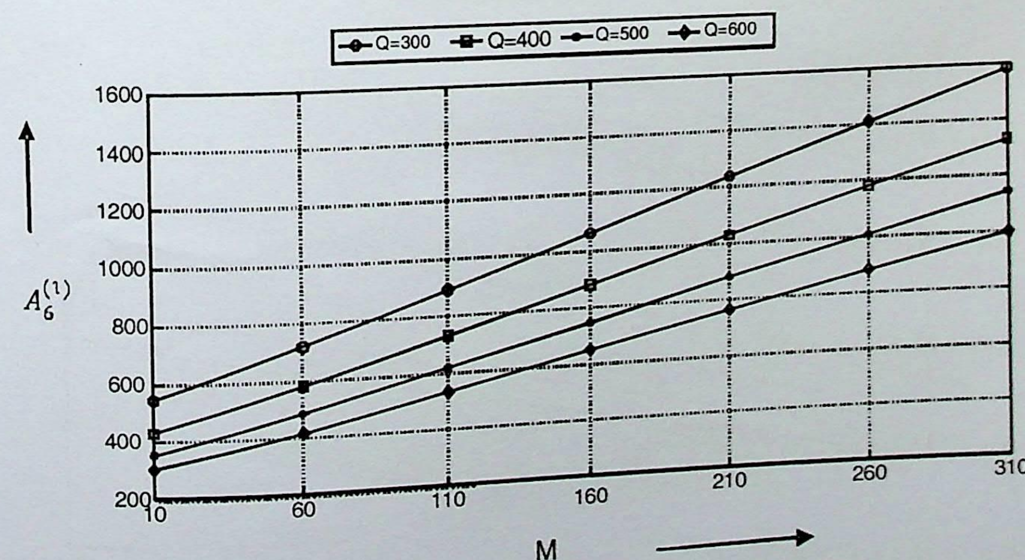


Figure-5: Graph between M and $A_6^{(1)}$ for Q=300, Q=400, Q=500, Q=600

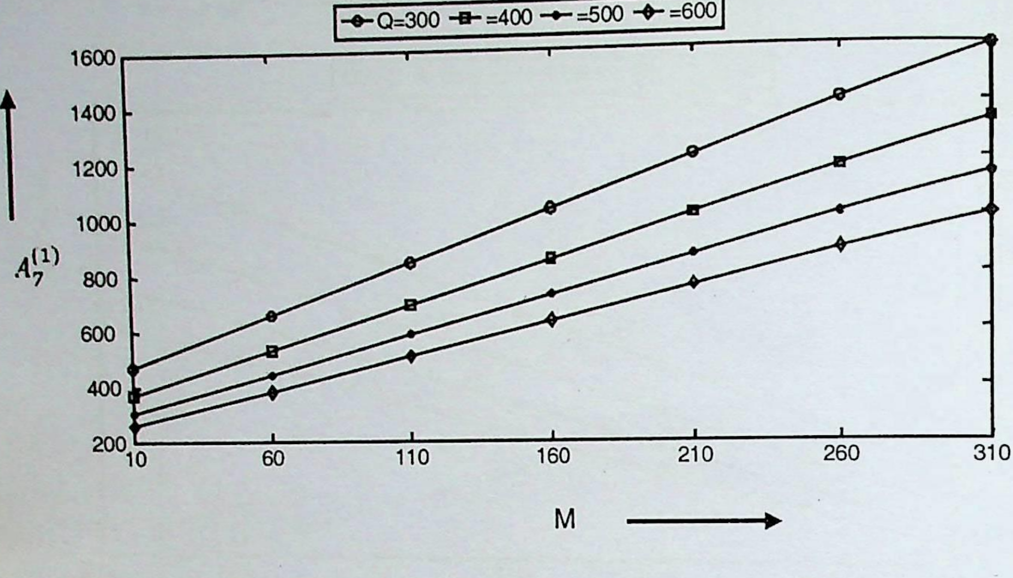


Figure-6: Graph between M and $A_7^{(1)}$ for Q=300, Q=400, Q=500, Q=600

Extended Hermite Polynomial of two vairable with application in Large Size of Animal Population. Now, let $z_n = H_n(x,y)$, where

$H_n(x,y) = n!\sum_{r=0}^{[n/2]} \frac{y^r x^{n-2r}}{r!(n-2r)!}$ Hermite polynomial.

$$n=1, z_1 = H_1(x,y) = 1$$

$$n=2, z_2 = H_2(x,y) = 2!\sum_{r=0}^{1} \frac{y^r x^{2-2r}}{r!(2-2r)!} = x^2 + 2y$$

$$n=3, z_3 = H_3(x,y) = 3!\sum_{r=0}^{1} \frac{y^r x^{3-2r}}{r!(3-2r)!} = x^3 + 6xy$$

$$n=4, z_4 = H_4(x,y) = 4!\sum_{r=0}^{2} \frac{y^r x^{4-2r}}{r!(4-2r)!} = x^4 + 12x^2y + 12y^2$$

$$n=5, z_5 = H_5(x,y) = 5!\sum_{r=0}^{2} \frac{y^r x^{5-2r}}{r!(5-2r)!} = x^5 + 20x^3y + 60xy^2$$

$$n=6, z_6 = H_6(x,y) = 6!\sum_{r=0}^{3} \frac{y^r x^{6-2r}}{r!(6-2r)!} = x^6 + 30x^4y + 180x^2y^2 + 120y^3$$

$$n=7, z_7 = H_7(x,y) = 7!\sum_{r=0}^{3} \frac{y^r x^{7-2r}}{r!(7-2r)!} = x^7 + 42x^5y + 420x^3y^2 + 840xy^3.$$

Example 2. Let $A_0^{(1)} = 1000, A_0^{(2)} = 2000, B_1 = 0.1, T_1 = T_2 = T = 0.01,$

$M_1 = M_2 = M = 10, 30, 50$

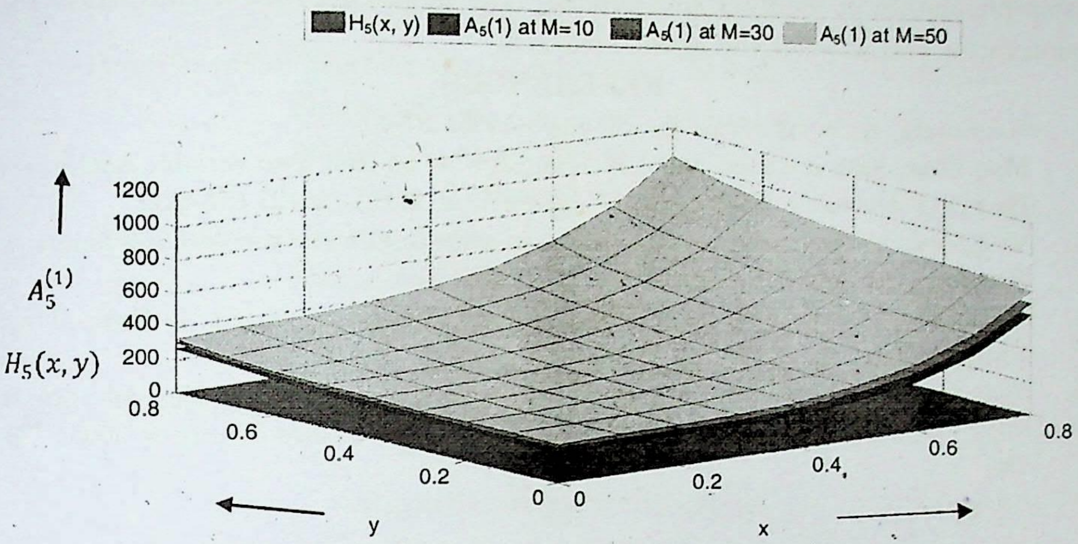


Figure-7: Graph between x, y, $A_5^{(1)}$ and $H_5(x,y)$ at M=10, 30, 50

Example 3. Let $A_0^{(1)} = 1000, A_0^{(2)} = 2000, B_1 = 0.1, T_1 = T_2 = T = 0.001,$

$M_1 = M_2 = M = 10, 40, 70$

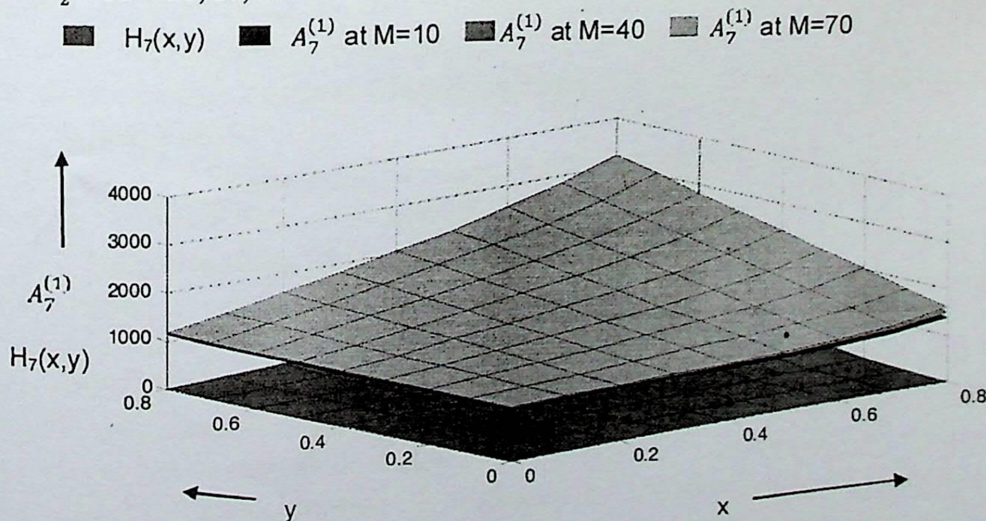


Figure-8: Graph between x, y, $A_7^{(1)}$ and $H_7(x,y)$ at M=10, 40, 70

Conclusion. The figures drawn depict three significant age groups and are self-explanatory. The pupulation under discussion is also under the influence of predators whose population does not change so rapidly in comparison to the main population of medium size animals. The migration rate is counted as the main factor and governs the growth pattern.

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SOLVING MULTI-ORDER LINEAR AND NON-LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS USING CHEBYSHEV WAVELET

By

R.S. Chandel

Department of Mathematics, Government Geetanjali Girls College,
Bhopal-462003, Madhya Pradesh, India
E-mail:rs_chandel2009@yahoo.co.in

Amardeep Singh

Department of Mathematics, Government Motilal Vigyan College,
Bhopal-462003, Madhya Pradesh, India

and

Devendra Chauhan

Department of Mathematics, IES, IPS Academy, Indore-452010,
Madhya Pradesh, India

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ABSTRACT

In this paper, we present a computational method based on Chebyshev wavelet for solving a multi order linear and non linear fractional differential equation. Chebyshev wavelet operational matrix of the fractional integration is derived and used to transform the differential equation to a system of algebraic equations. The validity and applicability of the technique is demonstrated by some examples and comparing the methodology with the known technique shows that the present approach is more efficient and more accurate.

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Key Words and Phrases : Chebyshev wavelet, fractional calculus, operational matirx, multi order linear and non-linear differential equation.

1. Introduction. In recent years, the use of fractional calculus deals with derivatives and integrals to an arbitrary order, in mathematical models, has become increasingly widespread [24,25]. Fractional calculus, as generalizations of classical integer-order differential equations are increasingly used to model some problems in fluid dynamic traffic [15], continnum and statistical mechanics [20], dynamics of interfaces between non particles and subtrates [7], non-linear oscillation of earthquakes [16], viscoelasticity [3], mechanics [27], signal processing [23], control theory [5], economics [4] and colored noise[21]. There has been significant interest in developing numerical schemes for the solution of fractional differential equation. Also there are different methods for solving multi order fractional differential

equations as a special kind of fractional differential equations [1,8,9,10,12,13,14,18, 19,20]. However, few papers have reported applications of wavelets in solving fractional differential equations [26,28].

Therefore the purpose of this paper to introduce the method of Chebyshev wavelet to solve multi orders fractional differential equation which include linear and non-linear differential equations. In this paper, the operational matrices of fractional order integrations are derived and a general procedure for forming these matrices is presented. In the proposed procedure, the Chebyshev wavelet expansions along with operational matrices of fractional order integrations are employed to reduce the multi order fractional differential equation to systems of non-linear algebraic equations. Illustrative examples are given to demonstrate the efficiency and applicability of the proposed method.

2. Basic Definitions and Notations.

2.1. Fractional Calculus. In this section, some necessary definitions and mathematical preliminaries of fractional calculus theory are given, which will be used further in this paper.

The Riemann-Liouville fractional integral operator I^{α} of order $\alpha>0$ on the usual Lebesgue space $L_1[a,b]$ is given by [24] as

$$\left(I^{\alpha}f\right)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-z)^{\alpha-1} f(z)dz, \tag{2.1.1}$$

$$\left(I^{0}f\right)(t)=f(t), \tag{2.1.2}$$

where $f \in C_{\mu}, \mu \geq -1$.

Its fractional derivative of order $\alpha>0$ is given by

$$\left(D^{\alpha}f\right)(t)=\left(\frac{d}{dt}\right)^n \left(I^{n-\alpha}f\right)(t), \text{ for } n-1<\alpha\leq n, \tag{2.1.3}$$

where n is an integer and $f \in C_1^n$.

Now by the Riemann-Liouvilles definition

$$\left.\begin{aligned} I^{\alpha}t^{\upsilon} &= \left\{\left(\Gamma(\upsilon+1)\right)/\left(\Gamma(\alpha+\upsilon+1)\right)\right\}t^{\alpha+\upsilon} \\ I^{\alpha}I^{\beta}f(t) &= I^{\alpha+\beta}f(t) \\ I^{\alpha}I^{\beta}f(t) &= I^{\beta}I^{\alpha}f(t) \end{aligned}\right\}, \tag{2.1.4}$$

where $\alpha,\beta \geq 0, t>0$ and $\upsilon>-1$.

There are many disadvantages of Riemann-Liouvilles derivatives when trying to model real world phennomena with fractional differential equations. Therefore we need to introduce a modified fractional differential operator proposed by Caputo [25]

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-z)^{n-\alpha-1} f^n(z)dz, (n-1 < \alpha \leq n) \tag{2.1.5}$$

where n is an integer $t>0$ and $f \in C_1^n$.

Caputo integral operator has a useful property,

$$I^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, (n-1 < \alpha \leq n), \tag{2.1.6}$$

where n is an integer $t>0$ and $f \in C_1^n$.

2.2. Chebyshev Wavelet. Wavelets constitute a family of functions constructed from dilations and translations of a single function called the mother wavelet $\psi(t)$. We have the following family of continuous wavelets in case the dilation parameter a and the translation parameter b vary continuously,

$$\psi_{a,b}(t) = |a|^{-1/2} \psi\left(\frac{t-b}{a}\right), \qquad a, b \in R, \qquad a \neq 0. \tag{2.2.1}$$

If we suppose parameter a and b to take discrete values as $a = a_0^{-k}, b = nb_0 a_0^{-k}, a_0 > 1$, $b_0 > 0$ for n and k positive integers, we have the following family of discrete wavelets

$$\psi_{k,n}(t) = |a_0|^{k/2} \psi(a_0^k t - nb_0), \tag{2.2.2}$$

where $\psi_{k,n}(t)$ forms a wavelet basis for $L^2(R)$. In particular, $\psi_{k,n}(t)$ forms an orthonormal basis, if $a_0=2$, $b_0=1$, that is

$$(\psi_{k,n}(t), \psi_{l,m}(t)) = \delta_{kl}\delta_{nm}.$$

The Chebyshev wavelets $\psi_{n,m}(t) = \psi(k, \hat{n}, m, t)$ have four arguments; $k \in N$, $n=1,2,\ldots,2^{k-1}$ and $\hat{n} = 2n-1$. m is the order of the Chebyshev polynomials of the first kind and t is the normalized time.

Chebyshev wavelets $\psi_{n,m}(t)$ are defined on the interval [0,1] as

$$\psi_{n,m}(t) = \begin{cases} 2^{k/2} \widehat{T}_m(2^k t - \hat{n}), & \frac{\hat{n}-1}{2^k} \leq t < \frac{\hat{n}}{2^k} \\ 0, & \text{otherwise} \end{cases}, \tag{2.2.3}$$

where $\widehat{T_m}(t)=\begin{cases}1/\sqrt{\pi}, & m=0\\ \sqrt{2/\pi}T_m(t), & m>0\end{cases}$

and $m=0,1,\ldots,M-1, n=1,2,\ldots,2^{k-1}$ where k is any positive integer and $T_m(t)$ are Chebyshev polynomial of the first kind of degree m, which are orthogonal with respect to the weighted function $\omega(t)=\frac{1}{\sqrt{1-t^2}}$ on the interval [-1,1] and $T_m(t)$ can be determined by the following recurrence formulae,

$$T_0(t)=1, T_1(t)=t, T_{m+1}(t)=2T_m(t)-T_{m-1}(t), \qquad m=1,2,3,\ldots .$$

Now in dealing with the Chebyshev polynomials the weighted function $\hat{\omega}=\omega(2t-1)$ has to be dialated and translated as

$$\omega_n(t)=\omega\left(2^k t-\hat{n}\right), \tag{2.2.4}$$

to get orthogonal wavelets.

A function $f(t)$ defined over [0,1) may be expanded as follows

$$f(t)=\sum_{n=0}^{\infty}\sum_{m\in z}C_{nm}\psi_{nm}(t), \tag{2.2.5}$$

by the Chebyshev wavelets, where

$$C_{nm}=\left(f(t),\psi_{nm}(t)\right)_{\omega_n}=\int_0^1\omega_n(t)\psi_{nm}(t)f(t)dt \tag{2.2.6}$$

In which (*,*) denote the inner product in $L^2_{\omega_n}[0,1]$. If the ifininte series in equation (2.2.5) is truncated, then it can be written as

$$f(t)\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}C_{nm}\psi_{nm}(t)=C^T\Psi(t), \tag{2.2.7}$$

where C and $\Psi(t)$ are $2^{k-1}M\times 1$ matrices given by

$$C=\left[C_{10},C_{11},\ldots,C_{1(M-1)},C_{20},\ldots,C_{2(M-1)},\ldots,C_{2^{k-1}0},\ldots,C_{2^{k-1}(M-1)}\right]^T \tag{2.2.8}$$

and

$$\Psi(t)=\left[\psi_{10},\psi_{11},\ldots,\psi_{1(M-1)},\psi_{20},\ldots,\psi_{2(M-1)},\ldots,\psi_{2^{k-1}0},\ldots,\psi_{2^{k-1}(M-1)}\right]^T. \tag{2.2.9}$$

Taking the collocation points as

$$t_i=\frac{2i-1}{2^k M}, \qquad i=1,2,\ldots,2^{k-1}M, \tag{2.2.10}$$

we define Chebyshev wavelet matrix $\phi_{m\times m}$ as,

$$\phi_{m\times m}=\left[\Psi\left(\frac{1}{2m}\right),\Psi\left(\frac{3}{2m}\right),\dots,\Psi\left(\frac{2m-1}{2m}\right)\right], \tag{2.2.11}$$

where $m=2^{k-1}M$.

For example, when M=3 and K=2, the Chebyshev wavelet is expressed as,

$$\phi_{6\times 6}=\begin{pmatrix} 2.2568 & 2.2568 & 2.2568 & 0 & 0 & 0\\ 1.0638 & 9.5746 & 18.0854 & 0 & 0 & 0\\ -2.4823 & 54.2562 & 201.7761 & 0 & 0 & 0\\ 0 & 0 & 0 & 2.2568 & 2.2568 & 2.2568\\ 0 & 0 & 0 & 1.0638 & 9.5746 & 18.0854\\ 0 & 0 & 0 & -2.4823 & 54.2562 & 201.7761 \end{pmatrix}$$

A function $k(x,t)\in L^2_{\omega_n}([0,1]\times[0,1])$ may be approximated as

$$k(x,t)=\Psi(x)^T K\Psi(t), \tag{2.2.12}$$

where K is $2^{k-1}M\times 2^{k-1}M$ matrix with

2.3 Operational matrix of the fractional integration. The integration of the vector $\Psi(t)$ defined in (2.2.9) can be obtained as $\int_0^t \Psi(t)dt\approx P\Psi(t)$ where P is the $2^{k-1}M\times 2^{k-1}M$ operational matrix for integration [17].

Our purpose is to derive the Chebyshev wavelet operational matrix of the fractional integration. We rewrite Riemann-Liouville fractional integration as follows

$$\left(I^\alpha f\right)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}*f(t), \tag{2.3.1}$$

where $\alpha\in R$ is the order of the integration, $\Gamma(\alpha)$ is the Gamma function and $t^{\alpha-1}*f(t)$ denotes the convolution product of $t^{\alpha-1}$ and $f(t)$.

Now if is expanded in the Chebyshev wavelet, as shown in the equation (11) the Riemann-Liouville fractional integration becoems,

$$\left(I^\alpha f\right)(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}*f(t)\approx C^T\frac{1}{\Gamma(\alpha)}\left\{t^{\alpha-1}*\Psi(t)\right\}. \tag{2.3.2}$$

Thus if $t^{\alpha-1}*f(t)$ can be integrated, then expanded in Chebyshev wavelets, the Riemann-Liouville fractional integration is solved via the Chebyshev wavelets.

Next, we define a m-set of Block pulse function (BPF) as

$$b_i(t)=\begin{cases}1, & i/m \le t < (i+1)/m \\ 0, & otherwise\end{cases} \tag{2.3.3}$$

where $i=0,1,2,...,(m-1)$.

The BPF has two useful properties which will be used further,

(i) Disjointness.

$$b_i(t)b_l(t)=\begin{cases}0, & i \ne l \\ b_i(t), & i=l\end{cases} \tag{2.3.4}$$

(ii) Orthogonality.

$$\int_0^1 b_i(\tau)b_l(\tau)d\tau=\begin{cases}0, & i \ne l \\ 1/m, & i=l\end{cases} \tag{2.3.5}$$

Similarly Chebyshev wavelet may be expanded into an m-term BPF as

$$\psi_m(t)=\phi_{m\times m}B_m(t), \tag{2.3.6}$$

where $B_m(t) \triangleq \left[b_0(t), b_1(t), ..., b_{m-1}(t)\right]^T$.

Kilicman and Al Zhour [18] have given the Block Pulse operational matrix of the fractional integration F^α as

$$\left(I^\alpha B_m\right)(t) \approx F^\alpha B_m(t),$$

where

$$F^\alpha=\frac{1}{m^\alpha}\frac{1}{\Gamma(\alpha+2)}\begin{pmatrix}1 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \xi_1 & \cdots & \xi_{m-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1\end{pmatrix}, \tag{2.3.8}$$

with $\xi_k=(k+1)^{\alpha+1}-2k^{\alpha+1}+(k-1)^{\alpha+1}$.

Now we derive the Chebyshev wavelet operational matrix of the fractional integration.

Let $\left(I^\alpha \Psi_m\right)(t) \approx P^\alpha_{m\times m}\Psi_m(t),$ (2.3.9)

where matrix $P^\alpha_{m\times m}$ is called the Chebyshev wavelet operational matrix of the fractional integration. Now on using equations (2.3.6) and (2.3.7), we have

$$\left(I^\alpha \Psi_m\right)(t) \approx \left(I^\alpha \phi_{m\times m}B_m\right)(t)=\phi_{m\times m}\left(I^\alpha B_m\right)(t) \approx \phi_{m\times m}F^\alpha B_m(t). \tag{2.3.10}$$

From equation (2.3.9) and (2.3.10), we get

$$P^{\alpha}_{m\times m}\Psi_m(t)=P^{\alpha}_{m\times m}\phi_{m\times m}B_m(t)=\phi_{m\times m}F^{\alpha}B_m(t). \tag{2.3.11}$$

Then, the Chebyshev wavelet operational matrix of the fractional integration $P^{\alpha}_{m\times m}$ is given by

$$P^{\alpha}_{m\times m}=\phi_{m\times m}F^{\alpha}\phi^{-1}_{m\times m}. \tag{2.3.12}$$

Therefore, we have found the operational matrix of fractional integration for the Chebyshev wavelet $P^{\alpha}_{m\times m}$.

The fractional integration of the function t was selecteds to verify the correctness of matrices.

That is because the fractional integration of the function $f(t)=t$ is easily obtain as follows, $(I^{\alpha}f)(t)=\dfrac{\Gamma 2}{\Gamma(\alpha+2)}t^{\alpha+1}$ which is easily used to compare the result obtained by the proposed method.

3. Applications and Results. In this section, we will use the Chebyshev wavelet operational wavelet operational matrices of the fractional integration to solve linear and non-linear fractional order differential equation. These examples are considered because closed form solutions are available for them of they have also been solved using other numerical schemes. This allows us to compare the results obtained using this scheme with the analytical solution or the solutions obtained using other schemes.

Example 1. Our first example deals with the homogeneous linear fractional differential equation

$$D^{\alpha}y(t)=-y(t),y(0)=1,0<t\leq 1 \text{ where } 0<\alpha\leq 1 \tag{3.1.1}$$

Let, $D^{\alpha}y(t)=K^T_m\Psi_m(t)$,

$$y(t)=K^T_mP^{\alpha}_{m\times m}\Psi_m(t)+[1,1,...,1]B_m(t)$$

Since $\Psi_m(t)=\phi_{m\times m}B_m(t)$,

$$y(t)=K^T_mP^{\alpha}_{m\times m}\phi_{m\times m}B_m(t)+[1,1,...,1]B_m(t).$$

Therefore by equation (3.1.1)

$$K^T_m\Psi_m(t)+K^T_mP^{\alpha}_{m\times m}\phi_{m\times m}B_m(t)+[1,1,...,1]B_m(t)=0. \tag{3.1.2}$$

This is a system of linear algebraic equation. We use MATLAB to solve equation (3.1.2). The exact solution of equation (3.1.1) is given by

$$y(t)=E_{\alpha}(-t^{\alpha}), \tag{3.1.3}$$

where $E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(ak+1)}$ is the Mittag-Leffler function of order α.

Numerical results for example 1 when $\alpha = 0.5, 0.75$ and 1 with comparison to ref. [22] are given in the table (1).

Table 1

t	α=0.5		α=0.75		α=1	
	y(Appro.)	y(exact)	y(Appro.)	y(exact)	y(Appro.)	y(exact)
0.0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.723854	0.723578	0.828250	0.828251	0.904837	0.904837
0.2	0.644794	0.643788	0.732486	0.732585	0.818732	0.818731
0.3	0.592179	0.592018	0.660340	0.660337	0.740816	0.740818
0.4	0.553728	0.553606	0.602023	0.602121	0.670319	0.670320
0.5	0.523256	0.523157	0.553606	0.553603	0.606529	0.606531
0.6	0.498126	0.498025	0.512289	0.512285	0.548811	0.548812
0.7	0.476605	0.476703	0.476557	0.476555	0.496586	0.496585
0.8	0.458149	0.458246	0.445294	0.445292	0.449328	0.449329
0.9	0.441954	0.442021	0.417684	0.417682	0.406570	0.406570
1.0	0.427611	0.427584	0.393106	0.393108	0.367079	0.367079

From table 1, it is clear that the approximate solutions of the equation (3.1.1) are in high agreement with the exact solutions for different values of α.

Example 2. The second example deals with the nonlinear fractional differential equation

$$D^\alpha y(t)=\left[y(t)\right]^2-f(t) \tag{3.2.1}$$

where

$$f(t)=\frac{2}{(t+1)^2}, y(0)=-2, t>0, 0<\alpha\le 1.$$

Let

$$D^\alpha y(t)=K_m^T\Psi_m(t)$$

then $y(t)=K_m^T P_{m\times m}^\alpha \Psi_m(t)-2[1,1,\dots,1]B_m(t)$.

Since $\Psi_m(t)=\phi_{m\times m}B_m(t)$,

$$y(t)=K_m^T P_{m\times m}^\alpha \phi_{m\times m}B_m(t)-2[1,1,\dots,1]B_m(t)$$

Let

$K_m^T P_{m\times m}^\alpha \phi_{m\times m}=[a_1,a_2,\dots,a_m]$ and using equation (2.3.3), we have

$$\left[y(t)\right]^2 = \left[a_1 b_1(t) + a_2 b_2(t) + \ldots + a_m b_m(t)\right]^2 - 2[1,1,\ldots,1]B_m(t)$$
$$= \left[a_1^2, a_2^2, \ldots, a_m^2\right]B_m(t) - 2[1,1,\ldots,1]B_m(t)$$

Now $f(t) = f_m^T \Psi_m(t)$ and $\Psi_m(t) = \phi_{m\times m} B_m(t)$

So $f(t) = f_m^T \phi_{m\times m} B_m(t)$. Hence by equation (3.2.1), we have

$$K_m^T \Psi_m(t) - \left[a_1^2, a_2^2, \ldots, a_m^2\right]B_m(t) + 2[1,1,\ldots,1]B_m(t) + f_m^T \phi_{m\times m} B_m(t) = 0 \quad (3.2.2)$$

This is nonlinear system of algebraic equations, here we use MATLAB to solve equation (3.2.2). The exact solution of equation (3.2.1) in case of $\alpha=1$ is given by

$$y(t) = -2/(t+1) \quad (3.2.3)$$

Table 2 shows the approximate solutions of the equation (3.2.1) for different values of α using the method of Chebshev wavelet operational matrix. The value of $\alpha=1$ is the only case for which we know that the exact solution is $y(t)=-2/(t+1)$ and our approximate solution of the equation (3.2.1) in this case is in high agreement with the exact solution.

Table 2. Numerical results for example 2 when $\alpha = 0.2, 0.4, 0.6, 0.8$ and 1.0

t	α=0.2	α=0.4	α=0.6	α=0.8	α=1.0
0.0	-2.00000	-2.00000	-2.00000	-2.00000	-2.00000
0.1	-1.57189	-1.58883	-1.56450	-1.73540	-1.81817
0.2	-1.48649	-1.48992	-1.53360	-1.59188	-1.66666
0.3	-1.42058	-1.41462	-1.44370	-1.48260	-1.53845
0.4	-1.36657	-1.35255	-1.36907	-1.39286	-1.42855
0.5	-1.32105	-1.29978	-1.30710	-1.31635	-1.33333
0.6	-1.28225	-1.25414	-1.25245	-1.24985	-1.25000
0.7	-1.24859	-1.21416	-1.20425	-1.19121	-1.17649
0.8	-1.21925	-1.17881	-1.16135	-1.13896	-1.11111
0.9	-1.19340	-1.14720	-1.12280	-1.09200	-1.05264
1.0	-1.170478	-1.11891	-1.08801	-1.04955	-0.99999

Example 3. Our third example covers the inhomogeneous linear equation

$$D^\alpha y(t) = -y(t) + \frac{1}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} + t^2 - t, \quad (3.3.1)$$

with $y(0) = 0, t > 0$ and $0 < \alpha \le 1$.

Let $f(t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} + t^2 - t,$

$$D^{\alpha}y(t) = K_m^T \Psi_m(t). \tag{3.3.2}$$

Then $y(t) = K_m^T P_{m\times m}^{\alpha} \Psi_m(t)$, $\Psi_m(t) = \phi_{m\times m} B_m(t)$.

Hence $y(t) = K_m^T P_{m\times m}^{\alpha} \phi_{m\times m} B_m(t)$. (3.3.3)

Now $f(t) = f_m^T \Psi_m(t)$ and $\Psi_m(t) = \phi_{m\times m} B_m(t)$,

Therefore $f(t) = f_m^T \phi_{m\times m} B_m(t)$. (3.3.4)

where f_m^T is a known constant vector.

Now making an appeal to (3.3.2),(3.3.3) and (3.3.4) from drive (3.3.1), we derive

$$K_m^T \Psi_m(t) + K_m^T P_{m\times m}^{\alpha} \phi_{m\times m} B_m(t) - f_m^T \phi_{m\times m} B_m(t) = 0. \tag{3.3.5}$$

This is a nonlinear system of algebraic equations, here we use the MATLAB function to solve the equation (3.3.5)

Th exact solution of equation (3.3.1) is given by $y(t) = t^2 - 1$

The linear equation(3.3.1) is solved by *K.* Diethelm using the fractional Adams-Bashforth-Moulton method [11]. Table 3 shows the exact solution by [11] and the approximate solution by our proposed method of equation (3.3.1) for different values of α. The results compare well with those obtained by Adams-Bashforth-Moulton method.

Table 3. Numerical results for example 3 when $\alpha = 0.5, 0.75$ and 1.0.

t	α=0.5		α=0.75		α=1	
	y(Appro.)	y(exact)	y(Appro.)	y(exact)	y(Appro.)	y(exact)
0.0	0.000000	0.000000	0.000000	.0.000000	0.000000	0.000000
0.1	-0.089820	-0.090000	-0.089894	-0.090000	-0.089925	-0.090000
0.2	-0.159791	-0.160000	-0.159856	-0.160000	-0.159956	-0.160000
0.3	-0.209818	-0.210000	-0.209876	-0.210000	-0.209965	-0.210000
0.4	-0.239851	-0.240000	-0.239913	-0.240000	-0.239976	-0.240000
0.5	-0.249894	-0.250000	-0.249917	-0.250000	-0.249974	-0.250000
0.6	-0.239963	-0.240000	-0.239972	-0.240000	-0.239986	-0.240000
0.7	-0.210026	-0.210000	-0.210015	-0.210000	-0.210012	-0.210000
0.8	-0.160089	-0.160000	-0.160066	-0.160000	-0.160022	-0.160000
0.9	-0.090175	-0.090000	-0.090102	-0.090000	-0.090052	-0.090000
1.0	-0.000142	-0.000000	-0.000109	-0.000000	-.0000049	-0.000000

Conclusion. In this paper, the Chebyshev wavelet is constructed and its operational matrix of the fractional integration is derived. Then a numerical method based on Chebyshev wavelets expansions together with these matrices are

proposed to obtain the numerical solutions of multi order fractional differential equations. The approximate solutions for some fractional differential equations are obtained by our method. The proposed method is very simple in implementation as it is computer oriented. Several examples are given to demonsdtrate the powerfulness of the proposed method and numerical results show that the method is very efficient and correct for the numerical solutions of linear and nonlinear fractional differential equations.

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SUMMABILITY SCHOOLS IN INDIA

By
Prem Chandra
Vikram University, Ujjain,
Email: premchandraujnn@yahoo.in

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ABSTRACT

Almost hundred years ago research in Summability Theory and Applications started in India (precisely, in South India). Slowly but surely, it started to attract the scholars to work in this field. For more than sixty years, this fascinating field of research remained popular among the scholars. The first aim of this paper is to introduce the Summability Schools in India, mostly in Central and North India, scholars and research topics on which researches were carried out their work at thesecentres of research. Second aim is to let the readers know the migration of the scholars from one centre to another mostly to establish a new centre for research.

2010 Mathematics Subject Classification : 01-02.
Key Words and Phrases : Survey article, Summability Schools in India.

Introduction. The present paper (survey article) is based on my two lectures. The first lecture, which I gave in the "National Conference on History of Mathematics" organized by Department of Mathematics, Kumaun University, Nainital(Uttarakhand) during October 13-16, 2000 and the second "U.N.Singh Memorial Lecture" was delivered in the "International Colloquim on Historyof Mathematical Sciences and Symposium on nonlinear analysis" organized by Indian Society for History of Mathematics at SSJ Campus, Almora, Kumaun University, Nainital, during May 16-19, 2011. The whole paper is divided into the following sections:

1. General Information regarding early researches in Summability Theory in India.
2. The first Summability School in the Central and North part of India.
3. Other Summability Schools and Scholars in the North and Central part of India in Summability Theory and their migrations.

1. General Informations regarding earlyresearches in Summability Theory in India. By Summability School, we mean a Department of Mathematics of a University or Institute or even College where the researches in summability theory and its applications have dominated over the researches of other branch/ branches of Mathematics. Through publications, I could locate the following places

where the researches in summability theory were carried out.

PLACES IN INDIA

1. Aligarh
2. Allahabad
3. Annamalainagar
4. Baroda(Vadodara)
5. Berhampur
6. Bhubaneswar
7. Bombay(Mumbai)
8. Cuttack
9. Gorakhpur
10. Jabalpur
11. Kanpur
12. Kurukshetra
13. Madras (Chennai)
14. Madurai
15. Rohtak
16. Roorkee
17. Sagar
18. Sambalpur
19. Ujjain
20. Varanasi
21. Waltair.

From publications, it appears that researches in summabilitytheory and applications in India particularly in South India are being done since long, precisely, not later than 1918, which is clear from the following publication of a senior most worker of this field (late) K.Ananda Rau, Professor of Mathematics, Madras Presidency College, Madras:

Proc. London Math. Soc.(2), 17(1918), 334-336.

T. Vijayraghavan andC.T.Rajagopal were among those pupils of K. Ananda Rau who were very much inspired by him. T. Vijayraghavan, who influencedRajagopal in his later career, was a senior college-mate in the Honours classes of Rajagopal. Rajagopal began his teaching career in the Annamalai University in 1930-31. Those days, a very few universities were having their own teaching departments especially in Mathematics, and therefore it was very much difficult to have a team work. It was 1950, when Ramanujan Institute of Mathematics was founded by AlagappaChettiar, a well-known educationist of that time, in the then Madras University, Madras. In 1951, T. Vijayraghavan took the charge of headship and later C.T.Rajagopal also joined the Ramanujan Institute of Mathematics. After that other people also got associated with Ramanujan Institute of Mathematics(RIM). Before we give a few names who worked at Madras, Waltair, Annamalainagar, Madurai and Bombay (Mumbai) in Summability Theory and Applications, we mention here a unique contribution in "RIESZ SUMMABILITY" which is due to K. Chandrasekharan and S. Minakshisundaram, a monograph published for the Tata Institute of FunndamentalResearch(TIFR), Bombay, by Oxford University Press in 1952.

NAMES OF SOME SCHOLARS

1. K.Ananda Rau, Madras (1918)
2. T.Vijayraghavan, RIM, Madras (1926)

3. V. GanapathyIyer, Madurai (1935)
4. V. Ramaswami, Madurai (1935)
5. C.T. Rajagopal, RIM, Madras (1935)
6. S. Minakshisundaram, Andhra University, Waltair (1936)
7. K. Chandrasekharan, TIFR, Bombay (1942)
8. K.S.K.Iyengar, RIM, Madras (1943)
9. M.S.Ramanujan, RIM, Madras (1956)
10. M.R.Parameswaram, Madurai (1957)
11. M.S.Rangachari, RIM, Madras (1961)

Note: Against each name, number indicates one of the years of his publications, not necessarily the first one.

2. The First Summability School in the Central and North part of India (Allahabad). It was July 1932, when B.N.Prasad (12 Jan 1899–18 Jan 1966) returned back to Allahabad from Europe after earning his degrees of Ph.D. and D.Sc. respectively from Liverpool(England) and Paris(France). He started building up at Allahabad a strong school of Research on Summability Theory in which he succeeded. In the North and Central part of India, it was the first Summability School.

Earlier, B.N.Prasad completed his M.Sc. in 1921 from Banaras Hindu University. On the advice of Ganesh Prasad, he joined the Department of Mathematics of Banaras Hindu University as a research scholar in July 1921 with Ganesh Prasad and later, after one year, he joined the same Department as Assistant Professor of Mathematics. But after the departure of Ganesh Prasad from Banaras Hindu Universityto Calcutta he could not continue at Banaras Hindu Universityfor more time and left it to join the Department of Mathematics of Allahabad Universityin 1924. And in 1929, he left Allahabad for England for intensive study of the works done in the theory of trigonometric series by W.H.Young, G.H.Hardy, J.E.Littlewood, E.C.Titchmarsh, H.Lebesgue, etc.. He worked with E.C.Titchmarshfor Ph.D. degree at Liverpool and later with celebrated French Mathematician Arnaud Denjoy for D.Sc.degree at Paris, which was awarded on the 4th June 1932 by the University of Paris.

After establishing a School of Summability at Allahabad, B.N.Prasad founded the Allahabad Mathematical Society in 1958 and began publishing "Indian Journal of Mathematics" to facilitate the workers for the research opportunities at this center of learning. This journal attained an international status very soon.

2.1. Scholars of the first generation of this School (Pupils of B.N.Prasad)

The following are the names of some scholars of the first generation of theSummability School headed by B.N.Prasad, that is, pupils of B.N.Prasad:

1. M.L.Mishra (retired from Sagar, M.P.)
2. P.L. Bhatnagar (retired from Shimla, H.P.)
3. U.N.Singh (1917-1989) (retired from University of Delhi)
4. J.A.Siddiqi (Canada)
5. S.M.Mazhar (Kuwait)
6. T.Pati (retired from Allahabad Universityand expired in 2008 July 02)
7. S.R.Sinha (1931-1985) (expired while in the service of Allahabad University)
8. D.P.Gupta (retired from M. L. N. Engg. College (now NIT), Allahabad; currently President of Allahabad Mathematical Society and is settled in Allahabad)
9. S.N.Bhatt (retired from Allahabad University; he is settled in Allahabad}
10. Tarkeshwar Singh (retired from M. L. N. Engg. College (now NIT; he is settled in Allahabad), Allahabad)
11. N.D.Mehrotra (retired, Allahabad)
12. Pramila Srivastava (retired from Allahabad University; she expired in 2008 April 12)
13. L.M.Tripathi (retired from Banaras Hindu University)
14. SulaxanaKumari Gupta (Gorakhpur)
15. G.D.Dixit (retired, Auckland, New Zealand).

Most of the first generation workers(pupils of B.N.Prasad) of this school migrated either to take active part in establishing the learning centres in various universities of India or abroad as indicated in the brackets of the above list. I will give some brief commentaries about some of these scholars. Presently, I give some observations about P.L.Bhatnagar and U.N.Singh.

P.L.Bhatnagar who turned out to be an applied Mathematician was primarily associated with this School of Summability established by B.N.Prasad. Under Prasad's supervision, he published the following papers on summability:

1. On the convergence of the conjugate series of the derived Fourier series, Proc. Nat. Acad. Sci. India, 7(1937), 116-122.
2. On the summability of the conjugate series of Fourier series, Proc. Ind. Acad. Sci. India, 7(1938), 85-90.
3. The absolute summability of the conjugate series of the derived Fourier series, Bull.Calcutta Math. Sco., 30(1938), 17-26.
4. On the Abel summability of the conjugate series of Fourier series, Bull. Calcutta Math. Soc., 31(1939), 31-44.

However, P.L.Bhatnagar found Plasma Theory more interesting than SummabilityTheory so he switched over to Astrophysics and Plasma Theory and worked with A.C.Banerji for his

(i) D.Phil. (Allahabad University, 1940)

(ii) D.Sc. (Allahabad University, 1974).

U.N.Singh, after completing Post-Graduation from Allahabad University in 1944, he joined the Summability School and completed his thesis entitled "Strong Summability of Trigonometric Series" and was awarded D.Phil. degree in 1949. Later in 1951, he went to Paris to work with famous Franch Mathematician S. Mandelbrojt. U.N. Singh was awarded D.Sc. degree in February 1954. After coming back to India, he changed his interest to Modern Mathematics.

2.2. Scholars of the Second and the Third Generationof this school:

1. S.N.Lal (retired from Banaras Hindu University): worked with S.N.Bhatt. He is settled in Varanasi.
2. Nand Kishore (retired from Berhampur University, Odisha): worked with S.R.Sinha.
3. Z.U.Ahmad (retired from Aligarh Muslim University): worked with T.Pati for (i) D.Phil. at Allahabad and for (ii) D.Sc. at Jabalpur.
4. H.P. Dikshit (retired from Jabalpur University): worked with T.Pati for (i) D.Phil. at Allahabad and for (ii) D.Sc. at Jabalpur.
5. J.P.Sharma: worked with D.P.Gupta, MLNR Engg.College(now NIT), Allahabad.
6. Lal Bahadur Singh: worked with S.N.Bhatt.
7. Vikramaditya Singh (retired from HRI, Allahabad): worked with S.R.Sinha for D.Phil. &D.Sc. at Allahabad. He is now settled in Allahabad.
8. UshaKakkar: worked with S.N.Bhatt.
9. S.S. Tripathi: worked with S.N.Bhatt.
10. Anand Prakash: worked with Pramila Srivastava.
11. Ashok Saxena: worked with S.R.Sinha.
12. K. Prasad: worked with S.R.Sinha.
13. Shyam Lal: worked with L.M.Tripathi at B.H.U. and joined Allahabad

University in 2001 and stayed from 28-04-2001 to 17-05-2004. During this period he supervised (i)Prem Narayan Singh (ii) Gopal Krishan Singh (iii) Vijay Kumar Tripathi (iv) Anjali Shrivastava (v) Ramaakant Yadav (vi)Ramashrya Singh Yadav.

2.3. Topics of Research on which researches were carried out at this centre (Allahabad) are:

(A)Dirichlet's Series (D.S.)

(i) Multiplication of Dirichlet series.

(ii) Multiplication of absolutely summable D.S.

(iii) The summability of the Dirichlet's product of summable series.

(iv) The abscissa of (a) Strong summability of D.S. (b)Absolute summability of D.S.

(v) Strong RieszSummability of D.S.

(B) Infinite Series

(i) Limitation Theorems for Absolute RieszSummability Method.

(ii) Tauberian Theorems for

(a) Absolute NörlundSummability Method

(b) Absolute Matrix Summability Method

(c) Absolute RieszSummability Method

(iii) Consistency Theorems:

(a) Consistency Theorems for RieszSummability

(b) The Second Theorem of consistency for the absolute RieszSummability

(c) Unified Theorem for consistency for the absolute Rieszsummability

(d) Consistency theorems for absolute Rieszsummability with indices.

(iv) Summability of Infinite series by

(a) Absolute Abel Summability

(b) Absolute CesàroSummability

(c) Absolute NörlundSummability

(d) Absolute RieszSummability

(e) Absolute RieszSummability with indices

(f) Strong RieszSummability

(g) Discontinuous Riesz means.

(v) Absolute Summability factors for Cesàro, Riesz and Nörlundsummability methods.

(C) Fourier series and allied series

(i) Summability of Fourier series by

(a) $\left(N, \frac{1}{n+1}\right)$ – and (b) (N, p_n) – mean.

(ii) Absolute Summability of Fourier series by Abel, Cesàro, Nörlund and RieszSummability methods.

(iii) Localization Problems for Absolute Summability of Fourier series and factored Fourier series for

(a) $|A|$-summabilitymethod

(b) $|C,k|(k>0)$ summabilitymethod

(c) $|N,p_n|$ – method

(d) $|R,\lambda_n,\alpha|(\alpha>0)$ method.

(D) Some Other Areas

(i) Gibbs Phenomenon for Nörlund means.

(ii) Convergence of Ultrasphericallacunary series

(iii) Local property of Absolute Summability of Laplace series and rth derived series of Fourier series

(iv) Summability and strong RieszSummability of orthogonal series

(v) Absolute NörlundSummability of Jacobi series

(vi) Strong Riesz means of orthogonal series

(vii) Bounded variation of power series.

3. Other Summability Schools and Scholars in the North and Central part of India and their migrations :

3.1. ALIGARH

J.A.Siddiqi, U.N.Singh and S.M.Mazhar were among the main workers in the theory of summability and applications in early days in the Department of Mathematics, Aligarh Muslim University (A.M.U.), Aligarh. They were migrated from the Summability School at Allahabad. These scholars could not stay more time at Aligarh but during their stay they trained a number of persons in this field. In 1958, U.N.Singh left Aligarh for Baroda as Professor of Mathematics to establish a new Department of Mathematics in M.S. Rao University of Baroda. J.A.Siddiqi left for Canada and settled there, he is now late. S.M. Mazhar also left Aligarh and settled in Kuwait, he is also late. In the meantime Z.U.Ahmad, a senior most pupil of T.Pati, joined Department of Mathematics of A.M.U., Aligarh.

There were a number of persons who worked in Summability Theory and Applications at this centre of research . Names of some of the scholars who worked and still working are:

1. J.A.Siddiqi(retired & expired in Canada)
2. U.N.Singh(retired from Univrsity of Delhi)
3. S.M.Mazhar(retired in Kuwait)
4. Sarfaraz Umar(retired from A.M.U., Aligarh)
5. Z.U.Ahmad (retired fromA.M.U. , Aligarh)
6. Niranjan Singh (retired from Kurukshetra)
7. A.H. Siddiqi (retired from A.M.U., Aligarh and is now settled in Aligarh)
8. Abrar Ahmad Khan
9. F.M. Khan (retired from A.M.U., Aligarh)
10. P.M.Abdul Rahiman
11. M.A. Khan
12. Q.A.Khan
13. A.H.Saifi
14. K.C.Varshney(retired from A.M.U., Aligarh)
15. H.H.Khan
16. Mohd. M.Rehman Khan

17. R.S.Khan
18. H.T.Jalal
19. Mursaleen.

3.2. VARANASI

Pramila Srivastava and L.M. Tripathi were pupils of B.N. Prasad. They left Allahabad to join the Department of Mathematics of Banaras Hindu University, Varanasi. Later, S.N. Lal, who worked with S.N. Bhatt, also left Allahabad to join the Department of Mathematics, Banaras Hindu University, Varanasi. They trained a number of students in Summability Theory. However, Pramila Srivastava came back to Allahabad in 1970 and retired from the Allahabad University, Allahabad, while L.M. Tripathi and S.N.Lal, continued to be in B.H.U., Varanasi, and retired from there as Professor of Mathematics. We now present a list of scholars (may be incomplete) in Summability Theory who worked in the Department of Mathematics of Banaras Hindu University, Varanasi.

(a) The pupils of L.M.Tripathi are:

1. G.C.Dvivedi 2. Kanhiya Prasad 3. A.P.Singh
4. K.N.Singh 5. Vaagesh Narayan Tripathi 6. V.P. Yadav
7. Shyam Lal 8. Shyam Narayan.

(b) The pupils of S.N.Lal are:

9. Sheo Ram Singh 10. Raj Deo Ram
11. Syaram 12. Kailash Nath Singh.

1.2.1. There had been another centre for research in Summability Theory located in VARANASI "Harishchandra P.G. College, Varanasi"now affiliated with Mahatma Gandhi KashiVidyapith, Varanasi.

In 1993, Shyam Lal of B.H.U. joined the Department of Mathematics of this college and supervised the following pupils in Summability Theory:

1. Sunita Verma 2. Ajay Pratap
3. K.N.Singh Yadav 4. Surendra Singh Yadav.

In November, 2007, Shyam Lal rejoined the Department of Mathematics of B.H.U. as Professor of Mathematics and supervised the following :

1. Dr. Binod Prasad Dhakkal 2. Dr. Harendra Prasad Singh.

3.3. GORAKHPUR

Miss Sulaxana Kumari Gupta, a pupil of late B.N.Prasad, left Allahabad University to join the Department of Mathematics, Gorakhpur University, Gorakhpur. Here, a few students got associated with her to do the research work in Summability Theory. One of her pupils was Babban Prasad Mishra, who after completing Ph.D. degree from Gorakhpur University went to England and worked with late B.Kuttner. SulaxanaKumari Gupta after her marriage to a Doctor resigned from the University, while B.P.Mishra, after coming back from England,

rejoined the University. During his period, B.P. Mishra, who expired in 2009 June 01, supervised the following students in Summability Theory at Gorakhpur:

1. Dinesh Singh
2. Sanjay Kumar Singh
3. Anupama Shrivastava
4. Mamta Rani Jaiswal
5. Sachchidanand Pathak
6. Gunjeshwar Shukla
7. Anand Prakash Shrivastava
8. Eklakh Ahmad Khan
9. Parthiweshwarji Pandey
10. Suyash Narayan Misra
11. Harish Chandra Singh
12. Dig Vijay Nath Pandey.

3.4. KANPUR

Kanpur is not much known for the workers in the Theory of Summability and its Applications. However, R.S.L. Shrivastava, Department of Mathematics, I.I.T., Kanpur contributed here to the Theory of Summability and its Application. He was associated with B.N.Prasad of Allahabad.

The other place in Kanpur, which I could locate is H.B.T.I. In fact, Shyam Lal, who was from Harishchandra P.G.College, Varanasi, joined H.B.T.I., Kanpur in August 1999 and stayed there till April 2001. He supervised the following students for their Ph.D. degree of Chhatrapati Sahu University, Kanpur:

1. Hare Krishan Nigam
2. Virendra Nath Tripathi
3. Purnima Yadav.

3.5. ROORKEE

In the Department of Mathematics of the University of Roorkee(now I.I.T.), the work in Summability Theory and Applications were initiated by O.P.Varshney. He worked in Summability Theory with M.L.Mishra (perhaps the first pupil of B.N.Prasad of Allahabad), Department of Mathematics, Saugar University(now Dr. H.S.GourUniversity), Sagar and left this university quite early to join the University of Roorkee. In September, 2001, this university was declared I.I.T.Roorkee because of the institution of National importance. O.P. Varshney worked there andeducated some scholars in Summability Theory and Applications. Those who have carried out or are still carrying their research work (in this institution) in Summability Theory and Applications are the following:

1. O.P. Varshney (Roorkee University)
2. M.L. Mittal (worked with Prof. O.P. Varshney, Roorkee Univ.)
3. G. Prasad (worked with Prof. O.P. Varshney, Roorkee Univ.)
4. Neeraj Bharadwaj (worked with Prof. O.P. Varshney, Roorkee Univ.)
5. Uaday Singh (I.I.T., Roorkee)
6. Vishnu Narayan Mishra (worked with M.L. Mittal, I.I.T., Roorkee)
7. Smita Sonker (I.I.T., Roorkee)
8. Shailesh Kumar Srivastava (I.I.T., Roorkee)
9. Mradul Veer Singh (I.I.T., Roorkee)

3.6. KURUKSHETRA

Niranjan Singh, who worked with S.M.Mazhar in the Department of Mathematics, Aligarh Muslim University, left Aligarh to join the Department of Mathematics, Kurukshetra University, Kurukshetra. He was a grand pupil ofB.N.Prasad. He along with his research students worked there in the Theory of Summability and its Applications to sequences and series. Niranjan Singh retired from KurukshetraUniversity as a Professor of Mathematics. Some of his pupils, who worked with him are:

1. Babu Ram 2. Rajiv Sinha 3. Pushpa Suri
4. Vinod Bhardwaj 5. K.M.Sharma.

3.7. R0HTAK

Babu Ram, who worked with Niranjan Singh in the Department of Mathematics, KurukshetraUniversity ,Kurukshetra, migrated to Rohtak to join the Department of Mathematics, M.D.University, Rohtak. He worked in Summability Theory and its Applications and supervised the following research students:

1. Renu Chug 2. Suresh Kumari 3. Satvinder Singh Bhatia
4. Navneet Hooda 5. Kulwinder Kaur 6. Jitendra Deep Kaur.

Babu Ram retired from M.D.University, Rohtok, as Professor of Mathematics, and in the meantime his first research pupil, Renu Chug became Professor of Mathematics in the M.D. University.

3.8. BARODA(Varodara)

It was 1958, when U.N.Singh joined The M.S.University of Baroda as Professor of Mathematics. B.S. Yadav, who was a student of U.N.Singh at Aligarh was so impressed that he came to Baroda to work with him. He got hisPh.D. degree on a thesis entitled "Operator Theory" possibly the first in India. Some of his works on Fourier series are referred to in a book: R.E.Edwards;Fourier Series(A Modern Introduction), Volume 1,SpringerVerlag, 1979.

In 1969, U.N.Singh left Baroda to join the Department of Mathematics of Delhi University as Professor of Pure Mathematics. After a few years, B.S.Yadav also left The M.S.University of Baroda to join the University of Delhi from where he retired.

V.M. Shah, who retired as a Professor of Mathematics from The M.S. University of Baroda,also worked with U.N.Singh on Lacunary Fourier series. There had been some other workers also at Baroda in Summability Theory and Applications, in particular, J.R.Patadia, a pupil of V.M.Shah. Thefollowing are a few names who worked at this centre of research:

1. U.N. Singh 2. B.S. Yadav 3. V.M. Shah 4. C.M. Patel
5. O.P.Goyal 6. J.R.Patadia 7. S.R.Agrawal 8. N.V.Patel.

3.9. SAGAR

Saugar University(now Dr. H.S.Gaur (Central) University) is the oldest University of M.P. The Department of Mathematics and Statistics of Saugar University was headed by M.L.Mishra, themost senior pupil of B.N.Prasad. Before retiring from this university, he educated a number of scholars in Summability Theory and Applications to Fourier series and allied series including the following:

1. Basudeo Singh (later migrated to Vikram University, Ujjain)
2. P.L. Sharma (Saugar University)
3. O.P. Varshney (later migrated toRoorkee University)
4. Sushil Kumar Shrivastava (Saugar University)
5. B.N. Sahney (migrated from Sagar to Canada).

After the retirement of M.L.Mishra, it was the turn of his pupilsBasudeo Singh and P.L.Sharma and their generation to educate scholars in this field. It is really difficult to name all the workers at this centre of research but we could manage to report the following:

6. B. Deokinandan (worked with Basudeo Singh and migrated to NCERT, Delhi)
7. S.B. Dubey (worked with Basudeo Singh retired from a college at Gwalior)
8. Mradula Shah (worked with Basudeo Singh and migrated to Jodhpur)
9. B. P. Katariya (worked with Basudeo Singh,Sagar)
10. P.D. Kathal (worked with P.L.Sharma, retired as Govt. College Principal)
11. R.K. Jain (worked with P.L.Sharma, Sagar University)
12. O.P. Rai (worked with P.L. Sharma, retired from M.A.C.T., Bhopal)
13. S.C. Kori (worked with P.L. Sharma, retired from Madhav Science College, Ujjain)
14. Ashok Ganguli (worked with R.K.Jain, retired from G.S.T.I., Indore)
15. R.S.Varma (worked with P.L.Sharma, retired from a college at Bilaspur, Chhattisgarh).

3.10. JABALPUR

After Allahabad, perhaps, Jabalpur had been one of the greatest centres of Summability Theory and Applications in the North and Central part of India. T.Pati, who laid down the foundation of the Department of Mathematics in the newly started Jabalpur University (later on renamed asR.D.University), Jabalpur, came from Allahabad University(Summability School). He stayed here for about eleven years and established a good centre of Summability Theory and Applications. He educated a good number of scholars in this field including the following and most of them migrated to different places in India and abroad:

1. Z.U. Ahmad (A.M.U., Aligarh)
2. H.P. Dikshit (R.D.University, Jabalpur)
3. G. Das (Utkal University, Bhubaneswar)

4. Ram Narayan Mohapatra (migrated to U.S.A.)
5. B.D. Malviya (migrated to Canada)
6. P. Panwar
7. V.P. Srivastava (Delhi University)
8. Prem Chandra (Vikram University, Ujjain)
9. D. Rath (Berhampur University, Odisha)
10. G. C. N. Kulshrestha (Govt. College Principal)
11. Satnam Narang (Tiwari)(Govt. College Principal)
12. E.C. Daniel (a college in Jabalpur)
13. Indu Karmohapatra (Sukla)(Sambalpur University)
14. M.L. Chandak (Agriculture University, Jabalpur)
15. Maya Mudiraj (Jabalpur University, Jabalpur)
16. S. Varma (a college in Nagpur)
17. S.V. Lokre (Nagpur)
18. K.N. Tripathi (a college in Mirzapur, U.P.)
19. V.S. Sai (Agriculture College, Rewa, M.P.)
20. P. Ray (Migrated from Jabalpur)
21. J. Banerji.

After the departure of T. Pati from Jabalpur back to Allahabad University, it was H.P.Dikshit who took over the charge of the Department of Mathematics. He also educated many workers in the Summability Theory and Applications and in some allied topics. Some of his pupils are:

22. Arun Kumar (R.D. University, Jabalpur)
23. B.L.Mishra (R.D. University, Jabalpur)
24. Namita Srivastava
25. Anil Dixit.

Arun Kumar, who worked with H.P. Dikshit, also supervised a number of scholars in the Summability Theory and Applications andin some allied topics. Some of the scholars who worked with him are:

26. D.P. Sahu (Raigarh, Chhattisgarh)
27. L.K. Govil (Govt. College, M.P.)
28. S.N. Singh
29. S.N. Dubey.

3.11. UJJAIN

BasudeoSingh who worked in Summability Theory and Applications in the Department of Mathematics and Statistics, Saugar University (Dr. H.S. Gaur University), Sagar, left the Saugar University to Paris for further studies where he worked with Laurent Schwartz and obtained D.Sc. degree. After coming back to India, he could not stay inSagar for a long time and joined Vikram University,

Ujjain, as Professor of Mathematics to establish a new Department(School of Studies in Mathematics and Statistics) and worked there till his retirement. In the meanwhile, G.S.Pandey, a pupil of D.P.Gupta, and Prem Chandra, who worked with T.Pati at Jabalpur, also joined the School of Studies in Mathematics and Statistics and educated several scholars in Summability Theory and Applications. Some of the names of the workers who were associated with this research centre are:

1. Basudeo Singh (Vikram University, Ujjain)
2. G.S. Pandey (Ph.D. & D.Sc. both from Vikram University, Ujjain)
3. N.V. Deshpandey (worked with G.S. Pandey and retired from D.A. Vishwavidyalaya, Indore)
4. Prem Chandra (Ph.D. from Jabalpur and D.Sc. from Vikram University,Ujjain)
5. Ashutosh Pathak (worked with Basudeo Singh,Ujjain)
6. Anurag Jaiswal (worked with Ashutosh Pathak, Neemuch)
7. Archana Beohar (worked with Ashutosh Pathak, Sonkachha, M.P.)
8. B.P. Khoti (worked with D.P.Gupta, Science College, Ujjain)
9. B.K. Beohar (worked with G.S.Pandey, Vikram University,Ujjain)
10. B.L. Jadia (worked with B.K. Beohar, retired from anadministrative post of M.P. Govt.)
11. B.P. Raghuwanshi (worked with Prem Chandra, Govt. Polytechnic, Ujjain)
12. H.C. Jain (worked with Prem Chandra, Shahdol, M.P.)
13. H.C. Sharma (worked with Prem Chandra , retired from R.B.S. College, Agra)
14. J.K. Gupta (worked with Ashutosh Pathak, Barwani, M.P.)
15. Jyoti Chaurasia (worked with Ashutosh Pathak, Itarsi, M.P.).
16. L.N.Jatwa (worked with Basudeo Singh and retired from Nani-Daman)
17. Mahesh Dalke (worked with Ashutosh Pathak, Mhow, M.P.)
18. Manisha Chaure (worked with Ashutosh Pathak, Khandwa, M.P.)
19. M.M. Sharma (worked with B.K.Beohar,Govt. Madhav ScienceCollege, Ujjain)
20. Neha Beohar (worked with Ashutosh Pathak, Ujjain)
21. Rajkumar (worked with Basudeo Singh and retired from Defense-Education)
22. R.C.Chaudhari (worked with D.P.Gupta. Govt.College, Dewas)
23. R.R.Gupta(worked with Prem Chandra , retired from Govt.College,Satna)
24. Sunita Gupta (worked with G.S.Pandey, Govt. College, Gwalior)
25. Saroj Joshi (orked with Prem Chandra ,Govt. Sci. College,Ratlam, M.P.)
26. Suresh Jain (worked with Ashutosh Pathak, Barwani, M.P.)
27. S.K.Sharma (worked with B.K.Beohar, Ujjain)
28. S.P. Yadav (worked with G.S.Pandey, Govt. Sci. College, Rewa)
29. T.P.Singh (worked with Prem Chandra, Govt. Sci. College, Rewa)
30. Tikam Singh (worked with D.P.Gupta, Engg. College, Ujjain)

31. Vandna Gupta (worked with Ashutosh Pathak, Ujjain)
32. VarshaKaranjgaokar(worked with Prem Chandra,Govt. Sci.Gollege, Raipur)
33. V.K.Gupta(worked with Ashutosh Pathak, Ujjain)
34. V.S.Yadav (worked with Prem Chandra, Agra)
35. Yusuf JalaliBafti (worked with Ashutosh Pathak, IRAN).

3.12. CUTTACK

The oldest centre of Summability Theory and Applications in Odisha (Orissa) was the Department of Mathematics, Revenshaw College, Cuttack, which was established by RamanathMohanty, the first and the most outstanding Mathematician of Odisha and the first President of the Orissa Math. Society, after coming back from London. Earlier, after completing Honours course in Mathematics at the graduate level, he went to Patna to complete Post Graduation in Mathematics from Patna University and served there as lecturer for a period of one year. Later he joined the Department of Mathematics, Revenshaw College, Cuttack. In 1945, he got the opportunity of proceeding to U.K. to work with J.E. Littlewood, and came in close contact with Mathematicians of world rank like G.H. Hardy, J.E. Littlewood, L.S. Bosanquet and others of the British School and within a couple of years he submitted his Ph.D. thesis to London University, under the supervision of L.S. Bosanquet, one of foremost analysts of the Hardy School. R. Mohanty spent the best period of his active academic life as teacher and researcher in Revenshaw College, Cuttack. His residence was a haven for scholars from different age groups includingBrajabehariPatanaik , T. Pati, G. Das and DiptyRath.

Mohanty's researches in London and thereafter weremostly concerned with Fourier Series. Several of his results led to many later investigations. For example, his criterion for the absolute convergence of Fourier series at a point is the first theorem of its kind. His investigations into the absolute Rieszsummability of the Lebesgue Fourier series at a point paved the way to further studies like those of T. Pati, N. Matsuyama (Japan), B.D. Malviya, Prem Chandra, etc.. He also collaborated with several workers like S.Izumi(Japan), B.Mishra, M. Nanda, J. Manmadha Rao, P.C. Rath, B.K. Ray etc. Some of those who wereeducated by him forresearch are:

1. B.K.Mishra	2. B. Mishra	3. B.Patra	4. G. Samal
5. J.N. Patnaik	6. M.K. Nayak	7. M. Nanda	8. N. Tripathi
9. S.Mohapatra	10. S.P. Misra .		

Further details may be found in Jour. Orissa Math. Soc., Volumes 12-15(1993-96).

3.13. BERHAMPUR

Perhaps, Department of Mathematics of Berhampur University was the first University centre for the research in Summability Theory and Applications in Odisha. This was headed by Nand Kishore who was one of the workers of the

second generation of B.N. Prasad's School of Summability at Allahabad. He educated a number of persons in Summability Theory and retired from there. In the meanwhile, D. Rath, a pupil of T. Pati, also joined the Department of Mathematics under the leadership of Nand Kishore. Rath, who was heading the Department and carrying out researches in Summability Theory, also retired from the University. The following are a few names of those who were associated with this centre of research:

1. Nand Kishore 2. D. Rath 3. U.N. Samantray 4. G.C.Hota
5. H.C.Jena 6. U.K.Mishra 7.Md. Abid 8. P.Padhey
9. Sakuntala Swain.

3.14. SAMBALPUR

When the University teaching Department of Mathematics was opened by the SambalpurUniversity atSambalpur, R.N.Mohapatra and InduKarmohapatra , who worked with T.Pati for their Ph.D. degrees at Jabalpur, joined the Department of Mathematics and began teaching in Mathematics and carried research in Summability Theory. However, Mohapatra could not continue inSambalpur University for more time and left Sambalpurin 1970 for abroad. Presently, he is a Professor of Mathematics in the University of Central Florida, Orlando, U.S.A.

After some time, Sambalpur University was shifted to BURLA, a place 10 kms. away from Sambalpur and GokulanandaDas, who was in Delhi University, joined theSambalpurUniversityas Professor and Head,Department ofMathematicthes.Earlier, G.Das started his teaching career from G.M.College, Sambalpur in 1960 and started meeting with T.Pati(at Jabalpur). He proceeded on leave from G.M.College ,Sambalpur to Jabalpur on C.S.I.R. Senior Research Fellowship for the period of two years to work with T. Pati for research work.

He completed Ph.D. work in 1965 and came back to Sambalpur. He was awarded Ph.D. degree in 1966 by UtkalUniversity, where he was registered for Ph.D. degree. From September, 1967 to July, 1969, G.Das went to England as a Commonwealth Scholar to work with L. S. Bosanquet at the University College, London, for Post-Doctoral Research work. He maintained his contact with T.Pati of Jabalpur who was his Ph.D. supervisor. After coming back to India, he joined G.M.College, Sambalpur but he could not stay there for more time and joined Delhi University as reader in Mathematics. In the meanwhile, he was awarded D.Sc. degree by the University of Jabalpur, in 1971.

In the Sambalpur University, he established the Department of Mathematics to carry out the researches in the fields of (i) Functional Analysis and Sequence spaces (ii) ErgodicTheory and (iii) Summability Analysis and Fourier series. However, G. Das did not stay for more time in this university also and migrated to

Utkal University, Bhubaneswar as Professor and Head, Department of Mathematics. As a result, only Indu Karmohapatra (now Mrs. Shukla) was left to continue the work in the Theory of Summability, from where she retired as Professor of Mathematics. Some of the names of the scholars who worked at Sambalpur (Colleges and University Department) are:

1. M.M. Nanda(G.M.College,Sambalpur)
2. G. Das (left forUtkal University,Bhubneswar)
3. R.N. Mohapatra (migrated to U.S.A.)
4. Indu Shukla (retired from this University).
5. S. Nanda (worked with G.Das, 1971, retired from IIT, Kharagpur)
6. S.L. Mishra (worked with G.Das, 1972)
7. S.K. Mishra (worked with G.Das, 1981)
8. S.K. Sahu (worked with G.Das, 1983)
9. B.K. Patel (worked with G.Das in Ergodic Theory, 1981)
10. P. Mohapatra (worked with G.Das, 1972)
11. P.C. Mohapatra (worked with G.Das, 1975)
12. N. Guru (worked with G.Das, 1985).

3.15. BHUBANESWAR

When G. Das joined the Department of Mathematics of Utkal University, Bhabaneswar, perhaps, there was no one who was working in the Summability Theory and Applications in the Department but soon after, a number of scholars joined this research centre which was developed by G.Das for research in (i) Functional Analysis and Sequence Space (ii) Summability Analysis and Fourier Series (iii) Fixed Point Theorems (iv) Probability Theory (v) Approximation Theory. However, somepersons who joined this research centre were from the different colleges of Utkal University, Bhubaneswar. G. Das, being greatly involved inresearch, collaborated with (1) L.S.Bosanquet (U.K.) (2) B.Kuttner (U.K.) (3) M.M. Nanda (4) Ram Narayan Mohapatra (5)V.P. Srivastava (6) B.K.Ray (7) R.C.Dash (8) J.P.Debata (9) Ellipse (10) Tulika Ghosh (11) N.Guru (12) S.K.Mishra (13) M.N.Mishra (14) Sangita Mishra (15) S.L.Mishra (16) S.K.Mohanty (17) M. Mohapatra (18) P. Mohapatra (19). P.C. Mohapatra (20) S.Nanda (21) Anusaya Nath (22) A.K. Ojha (23) K.C.Panda (24) B.K.Patel (25) Sumitra Patel (26) S. Patanaik (27) S.N.Pujari (28) A.K.S.S.V.K. Rao (29) K. Rao (30) B.N.B. Ray (31) Sudhakar Sahoo (32) S.K.Sahoo (33) PratimaSarangi (34) E. Savas (35) ManojaManjari Swain and many more.

From Das's remarkable and valuable contributions, I wish to mention a few of them. In [Proc. London Math. Soc.(3), 19(1969), 357-384], he developed a remarkable technique for a Tauberian Theorem which was adopted by a number

of workers in their research works including the present author. In 1979, a landmark and very important contribution on Summability Factor Problemswas published by him (cf. [Proc. London Math. Soc.(3),38(1979), 1-52] jointly with L.S.Bosanquet). In 1989, continuing furtherSummability Factor Problems, he [Indian Jour. pure and applied Math.,20(1989),347-385] developed Modified means with weights to handle factor problems quite effectively. Some very important contributions on the determination of new sublinear functional that generate Banach Limits have been published in three papers that appeared in Jour. London Math. Soc. in 1973. After retirement from the Department, he took over the charge of Vice-Chancellor of Utkal University and continued his research work and supervising a number of research scholars. Later, he took over the charge of the Chairman, Governing Council, Institute of Mathematics and Applications, Andharua, Bhubaneswar, and kept alive "Summability Theory and Applications". He is still attached with the research centre through his collaborators and research pupils and continuing research work. B.K. Ray, who worked with R. Mohanty and retired , is also associated with this research centre . I am giving below most of the names of the scholars who worked with G.Das in various fields in Utkal University till 2012:

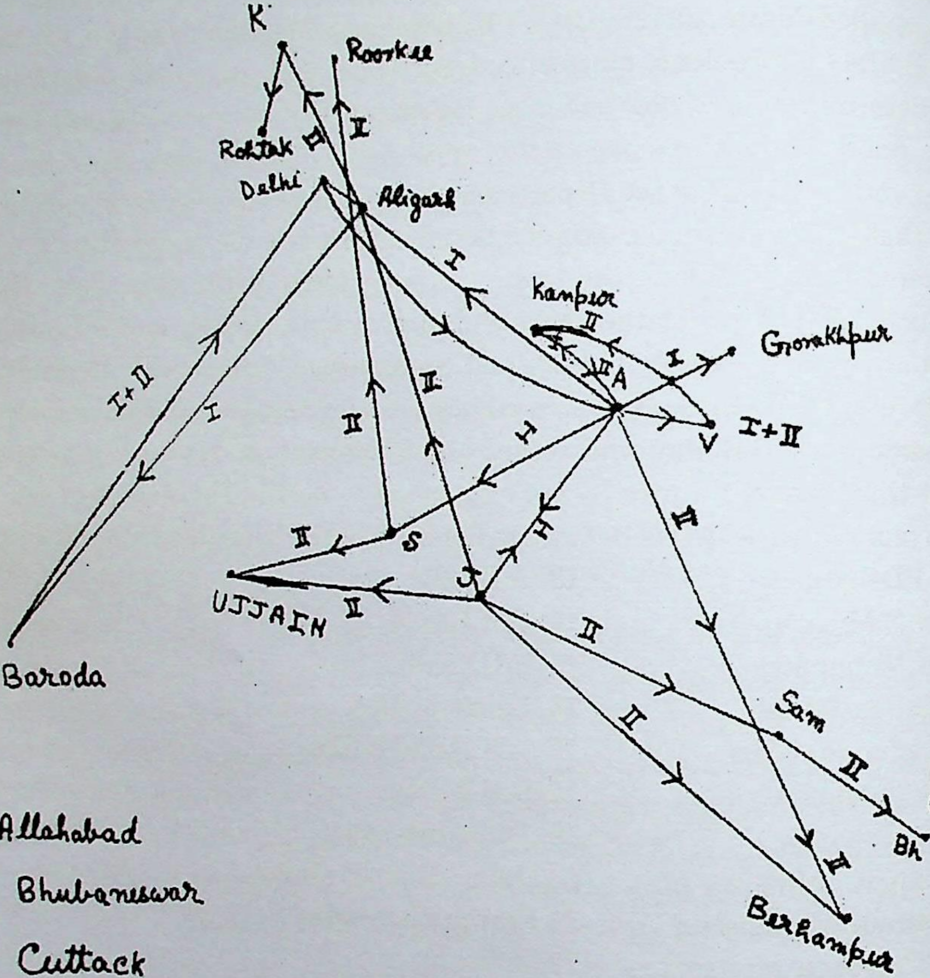
1. J.P.Debata
2. M.N.Mishra
3. M.Mohapatra
4. Tulika Ghosh
5. LaxmidharGiri
6. BijayMohanty
7. S. Mohanty
8. Sangita Mishra
9. Minati Mohapatra
10. Premlata Mohapatra
11. Arati Mishra
12. Anusaya Nath
13. A.K.Ojha
14. G.K.Panda
15. K.C.Panda
16. Sumitra Patel
17. A.K.S.S.V.K. Rao
18. B.N.B. Ray
19. Sudhakar Sahoo
20. ManojaManjari Swain
21. PratimaSarangi
22. M.Samal
23. E. Savas
24. Dr. Nandita Tripathi
25. Krishna Manjari Sahu

For further details, one may refer to Jour. Orissa Math. Soc., Vol.23-24(2004-2005). Finally we give a Road–map of the scholars who migrated from one research centre to another in India. To read the map, we indicate the meaning of notations used in the map.

(i) $\overset{\text{I}}{A \to B}$ and $\overset{\text{II}}{A \to B}$ indicate respectively migrations of pupils and grand pupils of B.N.Brasad from the research centre A to B.

(ii) I+II indicates pupils and grand pupils of B.N.Prasad.

ROAD-MAP



A = Allahabad

Bh = Bhubaneswar

C = Cuttack

J = Jabalpur

K = Kurukshetra

S = Sagar

Sam = Sambalpur

V = VARANASI

I = Pupil of Prof. B.N. Prasad

II = Grand pupil of Prof. B.N. Prasad

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5. *Jour. Orissa Math. Soc.,* (i) Volumes : **12-15** (Felicitation Volumes on the occasion of 85th birthday of Professor Ramanath Mohanty), 1993-1996. (ii) Volumes: **23-24** (Felicitation Volumes in honour of Professor Gokulanand Das on his 65th birthday), 2004-2005. (iii) Volume **27** (Professor Tribikram Pati Memorial Volume), 2008. (iv) Volume **30** (Professor Ramanath Mohanty Centenary Volume), 2011. (v) Volume **31**(2012).
6. *Sequences, Summability and Fourier Analysis*, Edited by D.Rath and S. Nanda, published by Narosa Publishing House Pvt. Ltd., New Delhi, 2005.

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MODELING THE EFFECTS OF INFECTED IMMIGRANTS AND PARTIAL FAILURE OF CONTROLLING STRATEGIES FOR HIV POSITIVE IN AIDS EPIDEMIC

By

Marindra Kumar Srivastava

Department of Mathematics, School of Management Sciences,
Technical Campus, Lucknow, 226016, Uttar Pradesh India
E-mail:mohitmarindra.23@yahoo.co.in

and

Abha Tenguria

Department of Mathematics, Government M.L.B. Postgraduate Girls
Autonomous College, Bhopal-462008, Madhya Pradesh, India

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ABSTRACT

In this paper, a mathematical model is formulated in order to study the effects of infected immigrants and partial failure of controlling strategies for *HIV* positives in heterosexual population of uniform promiscuous behavior. The model incorporates a newly launched controlling strategy, named as Detection, Treatmentation and Rehabilitation Program (*DTRP*) and its partial failureness due poor socio-economical conditions, illiteracy and myths of targeted population. We also introduce the immigration of infective males and infective females as functions, which are proportional to their corresponding density of sub-population. We show that there exists a positively invariant region where the model is epidemiologically and mathematically well-posed. The model does not exhibit infection-free state in general so we have investigated the criteria for existence of endemic Steady State along with positively invariant region of the model. We also compute reproductive rate and doubling time of infection using infection free steady state which is obtained in a specific case when inflow of infected immigrants in the population are restricted and the whole population occupied by susceptible immigrants. Finally, we determine local and global dynamics of these steady states of the system using stability theory and computer simulation and make conclusion.

2010 Mathematics Subject Classification : 37C75, 92B05

Key Words and Phrases : HIV Transmissions, Heterosexual Population, Asymptotic Stability, reproduction rates.

1. Introduction. Human Immunodeficiency Virus (HIV) has become one of the major public health problems in the World. HIV has killed more than 25 million people globally since its emergence in 1981, making it one of the most

destructive epidemics in recorded history. The infection continues to inflict a significant morbidity, mortality and social-economic and public health burden. Since vertical transmission of HIV-virus can occur during pregnancy, delivery or breast feeding. It is influenced by many factors, including maternal viral load and type of delivery. It is documentary evidenced that, about 20010 of the children infected with HIV virus develop AIDS symptoms in the first year of their lives and more of them die by the age of 4 years, more over up to 80 years of infected children develop HIV/AIDS symptoms at school age to adolescence age, CDCP,[6]. About 40% of all transmission cases of HIV virus result from mother to child transmission. So, HIV infection is a serious threat to future development of developing countries as India etc., where the level of literacy is low, the poverty level is very high and quality of health services is generally very poor.

Various preventative and therapeutic measures have been embarked upon, alarming at combating one of the greatest pandemics of modern times. As the current treatment therapy has been proven beyond reasonable doubt to reduce transmission, it is imperative to detect those who are infected and put them on treatment when they eligible, so they will be recovered from their infectiousness, (if it would be possible). Therefore, it is desirable to encourage voluntary testing that will increase case detection and also promote the researches toward desirable mediations or therapies. So it would be helpful in reducing the number of secondary infections in males and females by providing treatment to primary infective generations. Currently antiretroviral drugs (ARV) are known to be effective in lowering viral loads.

Mathematical modeling over the years have been useful in analyzing various disease dynamics, such as HIV/AIDS, Malaria and Tuberculosis and also play an important role in the better understanding of epidemiological patterns for disease control, (Makinde et. al.,[13]; Okosun et. al.,[14]; Waxman et.al.,[18]; Diekmann et.al.,[7]; Gielen et. al.,[9]). From the initial models of Anderson et al. [2] various refinements have been added into modeling frameworks, and specific issues have been addressed by researchers, (Perelson et.al.,[15]; Cai el. al.,[5]; Tripathi et.al.,[16]).

In recent years, numerous mathematical model have been extensively developed, to explore theoretical aspects of HIV/AIDS dynamics at population and cellular level, (Ferreira et. al.,[8]; Hsieh et.al.,[10]; Hsieh et.al.,[11]). Still most of these models ignore several serious facts related to heath challenges in population, where HIV/AIDS exists. In particular, Busenberg et.al.,[4], discussed a variety of diseases that transmit both horizontally and vertically, and gave a comprehensive survey of the formulation and the mathematical analysis of compartmental models

that also incorporate vertical transmission. Brauer et.al.,[3], considered models for disease with vertical transmission with non-linear population dynamics and finite carrying capacity, and analyzed the stability of equilibriums in the special case in which the overall birth rate does not depend on infective population size. Li M. Y. et.al.,[12], proposed a model for an infectious disease that spreads in the host population through both horizontal and vertical transmission.

A little attention has been paid to study the role of vertical transmission in HIV/AIDS models. Agarwala et.al.[1] ,developed a density dependent HIV transmission model for Canadian population by taking into account the vertical transmission and by using simple mass action type interaction. Also Tripathi et.al.,[17], examined a similar model to Agarwala et.al.,[1] with vertical transmission but did not consider infected immigrants coming into the community. The major difference between past frame work done and our study given in this chapter is that they do not consider HIV immigrants and also do not visualize & evaluate the best strategies that would able to suppressed HIV/AIDS in any given targeted population. And we considered not only HIV positive immigrants but also their inflow, in infection circulation, is proportional to the density of their corresponding sub population, which is become an important aspect of HIV/AIDS in developing countries.

In this chapter, we formulated a deterministic model in the light of above considerations. The model also incorporating visualization and evaluation of a newly launched controlling strategy, named as Detection, Treatmentation and Rehabilitation Program (DTRP) and its partial failureness due poor socio-economical conditions, illiteracy and myths of targeted population. Thus, this study is proposed to assess the effect of infected immigrants and partial failure of controlling strategies for HIV positives in the distribution dynamics of HIV /AIDS at heterosexual setting.

2. The Mathematical Model. Let at time t, $P(t)$ is the targeted heterosexual population for our study. We divided this population into five sub-classes, which are denoted as $S_1(t), I_1(t), S_2(t), I_2(t)$, are known as susceptible males, infective males, susceptible females and infective females respectively. The fifth sub-class denoted as $D(t)$ and visualized, for those males and females in targeted population, who were detected as HIV positive and then got treatment, when they were eligible by providing such therapies/medications or both, which would able to recover their HIV positiveness and finally they availed rehabilitation policy according to a newly launched controlling strategy, named as Detection, Treatmentation and Rehabilitation Program (DTRP). In the model, $g_2D(t)$ is the

proportion of those individuals, who were successfully detected and got treatment, when they eligible. But they could not received effective rehabilitation, which may be happened due to the scarcity of proper education, employment and also of many mythological issues appeared in their life as happened in developing countries, like India. So, they are supposed again as susceptible to infection circulation. And is the small proportion of those individuals, who were detected, got treatment and successfully availed rehabilitation. So they are emigrated from infection circulation. We assumed that susceptible male and female become HIV positive following contacts with the HIV infected female and male are at rate k. At instant of time t, a constant inflow of susceptible males and females in targeted population are at rates θ_1 and θ_2 respectively. We also introduced the immigration of infective males and infective females as functions, which are proportional to their corresponding density of sub-population and defined as $f_1(I_1)=\delta_1\pi I_1$ and $f_2(I_2)=\delta_2\pi I_2$. Where π is proportionality constant. In the model we ignored birth process in susceptible individuals because it is a sexually transmitted infection. Since, the individuals of $D(t)$ population are exposed, so they supposed to be incapable for producing their children. In the model, pb', qb', r and r' are rates of same sense that we have defined in previous modeling. These above assumption make our model different to other model have studied on the field.

The dynamics of infection is assumed to be governed by the following system of non-linear ordinary differential equations under the above assumptions:

$$\left.\begin{aligned}
\dot{S}_1 &= \theta_1 + pb'\beta(I_1+I_2) + hg_1D - kS_1I_2 - rS_1\\
\dot{I}_1 &= \delta_1\pi I_1 + qb'l(I_1+I_2) + kS_1I_2 - (r'+c)I_1\\
\dot{S}_2 &= \theta_2 + pb'(1-\beta)(I_1+I_2) + (1-h)g_2D - kS_2I_1 - rS_2\\
\dot{I}_2 &= \delta_2\pi I_2 + qb'(1-l)(I_1+I_2) + kS_2I_1 - (r'+c)I_2\\
\dot{D} &= c(I_1+I_2) - (r+g_1+g_2)D
\end{aligned}\right\}, \qquad \text{...(2.1)}$$

with initial conditions

$$\left.\begin{aligned}
&S_1(0)=S_{10}>0, I_1(0)=I_{10}\ge 0, D(0)=D_0\ge 0\\
&S_2(0)=S_{20}>0, I_2(0)=I_{20}\ge 0\\
&p+q=1, P(t)=S_1(t)+I_1(t)+S_2(t)+I_2(t)+D(t)
\end{aligned}\right\}. \qquad \text{...(2.2)}$$

The total population changed is

$$\dot{P}=(\theta_1+\theta_2)+(\pi+b'-r')(I_1+I_2)-r(S_1+S_2)-(r+g_1+g_2)D \quad . \qquad \text{... (2.3)}$$

Here c is the rate at which infective males and females are acknowledge into D(t) population due to preventive strategies or/and treatment therapies.

$\delta_1, l, \beta, (1-\beta), (1-l), \delta_2$ are the probabilities of being male and female respectively, so that $\delta_1 + \delta_2 = 1$. The flow diagram of model (2.1) is given in Fig. (1).

3. Positive and Invariant Region. Let $r_1 = \min\{r, (r + g_1 + g_2)\}$ then the equation (2.3) becomes as

$$\dot{P} = (\theta_1 + \theta_2) + (\pi + b')(I_1 + I_2) - r_1 P - (r' + r_1)(I_1 + I_2) \cdot \qquad \text{... (3.1)}$$

By using differential inequality, above equation become as

$$\dot{P} \le (\theta_1 + \theta_2) + (\pi + b')(I_1 + I_2) - r_1 P,$$

which implies that $\overline{P} = \dfrac{(\pi + b')(\overline{I_1} + \overline{I_2}) + (\theta_1 + \theta_2)}{r_1}$.

Here $\left(\overline{I}_1 + \overline{I}_2\right) = \left(S_1^* + S_2^*\right)\left[1 + \dfrac{\pi + qb' - (r' + c)}{k}\right]$,

$S^0 = S_1^0 + S_2^0 =$ Initial susceptible males and females at time $t=0$.

From (2.1), we get

$$\overline{D} = c(\overline{I_1} + \overline{I_2}) / r + g_1 + g_2 \cdot$$

Therefore from (3.1), we derive

$$\dot{P} + r_1 P = (\theta_1 + \theta_2) + [(\pi + b') - (r' + r_1)]\left[1 + \frac{\pi + qb' - (r' + c)}{k}\right](S_1^0 + S_2^0). \qquad \text{... (3.2)}$$

Hence complete solution of (3.1) is

$$P(t) \le P(0)e^{-r_1 t} + \frac{1}{r_1}\left[(\theta_1 + \theta_2) + [(\pi + b') - (r' + r_1)]\left[1 + \frac{\pi + qb' - (r' + c)}{k}\right](S_1 + S_2)\right](1 - e^{-r_1 t}). \qquad \text{... (3.3)}$$

Here $P(0)$ is the total number of population at initial level of study.

Thus it is suffices to consider the solution in the region

$$\Omega = \left\{\left(P, (I_1 + I_2), D : 0 \le P \le \overline{P}, 0 \le (I_1 + I_2) \le \left(\overline{I}_1 + \overline{I}_2\right), 0 \le D \le \overline{D}\right)\right\}$$

under the condition $(k + \pi + qb') > (r' + c)$.

Henceforth, Ω is positively invariant region of attraction for our model,

where all solutions remain in Ω, for all time. So it is sufficient to consider that the dynamics of model system (2.1) is considered to be epidemiologically and mathematically well posed in the region Ω.

4. Steady State Analysis. The endemic steady state of the model can be derived from the following set of equations

$$\left.\begin{aligned} &\theta_1 + pb'\beta(I_1 + I_2) + hg_1 D - kS_1 I_2 - rS_1 = 0 \\ &\delta_1 \pi I_1 + qb'l(I_1 + I_2) + kS_1 I_2 - (r' + c)I_1 = 0 \\ &\theta_2 + pb'(1-\beta)(I_1 + I_2) + (1-h)g_2 D - kS_2 I_1 - rS_2 = 0 \\ &\delta_2 \pi I_2 + qb'(1-l)(I_1 + I_2) + kS_2 I_1 - (r' + c)I_2 = 0 \\ &c(I_1 + I_2) - (r + g_1 + g_2)D = 0 \end{aligned}\right\} \quad \ldots (4.1)$$

Endemic Steady State $E^*(S_1^*, I_1^*, S_2^*, I_2^*, D^*)$:

$$S_1^* = \left[\frac{\theta_1 + (a_1 + \delta_1\pi - e)I_1^* + a_1 I_2^*}{r}\right],$$

$S_1^* > 0$, iff $I_1^* > 0, I_2^* > 0$ and $\delta_1 \pi > e$,

Where $a_1 = \left[(p\beta + ql)b' + \dfrac{hg_1 c}{r + g_1 + g_2}\right], e = (r' + c)$.

$$S_2^* = \left[\frac{\theta_2 + q_2 I_1^* + a_2 + (\delta_2\pi - e)I_2^*}{r}\right],$$

$S_2^* > 0$, iff $I_1^* > 0, I_2^* > 0$ and $\delta_2 \pi > e$.

Where $a_2 = \left[\{p(1-\beta) + q(1-l)\}b' + \dfrac{(1-h)g_1 c}{r + g_1 + g_2}\right],$

$$D^* = \left[\frac{c(I_1^* + I_2^*)}{(r + g_1 + g_2)}\right],$$

$D^* > 0$, iff $I_1^* > 0, I_2^* > 0$.

Now, we investigate criteria for positive steady state E^* to exist. We again use the set of equations (4.1). Firstly, a isocline in I_1^* and I_2^* from set of equation (4.1), we obtain

$$a\left[a_3 - (1-\delta_1)\pi\right]I_1^* + \left[a_3 - (1-\delta_2)\pi\right]I_2^* = (\theta_1 + \theta_2), \quad \ldots (4.2)$$

which implies that

$I_1^* \to \hat{I}_1$ when $I_2^* \to 0$ and $I_2^* \to \hat{I}_2$ when $I_1^* \to 0$.

Here $\hat{I}_1 = \left[\dfrac{\theta}{a_3 - (1-\delta_1)\pi}\right], \hat{I}_2 = \left[\dfrac{\theta}{a_3 - (1-\delta_2)\pi}\right],$

where $\theta = \max(\theta_1, \theta_2); a_3 = \left(\dfrac{g_1 c}{r + g_1 + g_2}\right).$

We can also obtain $\dfrac{dI_1^*}{dI_2^*} = -\left[\dfrac{a_3 - (1-\delta_2)\pi}{a_3 - (1-\delta_1)\pi}\right],$

$\dfrac{dI_1^*}{dI_2^*} > 0$, iff $a_3 > (1-\delta_1)\pi$ and $a_3 > (1-\delta_2)\pi$.

Hence, we find that I_1^* is decreasing function of I_2^*. Again, second Isocline in I_1^* and I_2^* from set of equation (4.1), we get

$$b_1(I_1^*)^2 + b_2(I_2^*)^2 + b_3(I_1^* \cdot I_2^*) + b_4 I_1^* + b_5 I_2^* = 0. \quad \text{... (4.3)}$$

Here

$b_1 = \left[\dfrac{a_2 + \delta\pi - e}{r}\right]; b_2 = \left(\dfrac{a_1}{r}\right), b_3 = \left[\dfrac{a_1 + a_2 + \delta_1\pi - e}{r}\right]$

$b_4 = \left[\pi + qb' + \dfrac{k\theta_2}{r} - e\right], b_5 = \left[\pi + qb' + \dfrac{k\theta_1}{r} - e\right],$

which implies that

$\left(b_1(I_1^*)^2 + b_4 I_1^*\right) = 0$ when $I_2^* = 0$,

such that $I_1^*\left[b_1 I_1^* + b_4\right] = 0$,

so that $I_1^* \to \tilde{I}_1 > 0$ when $I_2^* \to 0$.

Here $\tilde{I}_1 = (b_4/b_1) > 0$, iff $b_4 > 0$ and $b_1 > 0$,

where

$b_4 > 0 if \left(\pi + qb' + \dfrac{k\theta_2}{r}\right) > e$ and $b_3 > 0 if\,(a_2 + \delta_2\pi) > e$

and

$b_2(I_2^*)^2 + b_5 I_2^* = 0$ when $I_1^* = 0$,

such that $I_2^*[b_2 I_2^* + b_5] = 0$ so that $I_2^* \to \tilde{I}_2$ when $I_1^* \to 0$.

Here $\tilde{I}_2 = (b_5 / b_2) > 0$ iff $b_5 > 0$ and $b_2 > 0$,

where $b_5 > 0$ if $(\pi + qb' + k\theta_1 / r) > e$ and $b_2 > 0$ if $a_1 > 0$ and $r > 0$.

We can also obtain

$$\frac{dI_1^*}{dI_2^*} = -\left[\frac{(\pi + qb' + k\theta_1 / r - e) + (2a_1 / r) I_2^* + b_3 I_1^*}{2b_1 I_1^* + \pi + qb' + k\theta_2 / r - e}\right] < 0,$$

where $\pi + qb' + k\theta_1 / r > e, \delta_1 \pi > e$ and $\pi + qb' + k\theta_1 / r > e, \delta_2 \pi > e$.

Hence, we find that I_1^* is decreasing function of I_2^*.

From above we see that two isoclines (4.2) and (4.3) will be interested provided that $\tilde{I}_1 > \hat{I}_1$ and $\tilde{I}_2 > \hat{I}_2$. Hence with the above considerations, equation (4.2) represents I_2^* as decreasing from $\hat{I}_2$ and equation (4.3) I_2^* decreasing from $\hat{I}_2$. Therefore two isoclines must intersect provided that intersection value and then positive steady state E^* exists in $I_2^* - I_1^*$ plane, see Fig. (2).

5. Computation of Basic Reproductive Rate. In order to achieve infection free steady state, it is expected that the whole population is to be occupied by susceptible males and susceptible females and other sub-populations go to extinction after some time. So, we assume that at initial stage, when $\pi = 0$, the population $P \approx P^* = [\theta_1 / r + \theta_2 / r] = \theta / r$, here $\theta = \max\{\theta_1, \theta_2\}$, which is maximum of the recruitment of susceptible males and females.

From model (2.1) we get

$$I = I \ \exp[\{\pi + qb' + k\theta / r - (r' + c)\} t]. \qquad \ldots (5.1)$$

If $[\pi + qb' + \frac{k\theta}{r} - (r' + c)] < 0$ then the number of infective die out exponentially with time, then there is no infection at infinite time, which may implie that $R_0 < 1$.

Here $R_0 = \frac{\pi}{(r' + c)} + \frac{qb'}{(r' + c)} + \frac{k\theta}{r(r' + c)}$ = Basic Reproductive rate of model (2.1.1),

so that (5.1) can be rewritten as $I = I^0 \exp((R_0 - 1) / t)$. ... (5.2)

Here $\tau = 1/(r' + c)$ = Mean time during which males and females being infectious and $I^0 = \max\{I_1^0, I_2^0\}$.

Let $t = t_d$ and $I = 2I^0$ then (5.2) gives

$2I^0 = I^0 \exp\left((R_0 - 1)/\tau\right).t_d$,

which implies that

$t_d = \tau \ln 2/(R_0 - 1)$ = Doubling time of infection of model (2.1.1).

6. Local and Global Stability Analysis. To determine the local stability of endemic steady state E^*, the following variation matrix M_{E^*} is computed around the steady state E* :

$$M_{E^*} = \begin{bmatrix} -(kI_2^* + r) & pb'\beta & 0 & pb'\beta - kS_1^* & hg_1 \\ kI_2^* - r & \delta_1\pi + qb'l & 0 & qb'l + kS_1^* & 0 \\ 0 & pb'(1-\beta) - kS_2^* & -(kI_1^* + r) & pb'(1-\beta) & (1-h)g_1 \\ 0 & qb'(1-l) + kS_2^* & kI_1^* - r & \delta_2\pi + qb'(1-l) & 0 \\ 0 & c & 0 & c & -(r + g_1 + g_2) \end{bmatrix} \quad \dots (1)$$

The characteristics equation of M_{E^*} is

$$f(x) = x^5 + A_1\, x^4 + A_2\, x^3 + A_3\, x^2 + A_4\, x + A_5 = 0 \quad \dots (2)$$

Here, the values of A_i (for every i=1, 2, 3, 4, 5) are given in ***Appendix***.
Thus, by Routh-Hurwitz criteria, E*is locally asymptotically stable as it can be seen for

$A_i > 0, (\forall i = 1, 2, ..., 5)$; $A_1 A_2 A_3 > A_3^{\,2} + A_1^{\,2} A_4$ and

$(A_1 A_4 - A_5)(A_1 A_2 A_3 - A_3^{\,2} - A_1^{\,2} A_4) > A_5 (A_1 A_2 - A_3)^2 + A_1 A_5^2$

along with $[\pi + qb' + k\theta/r] > (r' + c)$.

Now we have given the conditions for global stability of endemic steady state of model (2.1) in the form of following

Theorem. If the endemic steady state $E*$ exists then it is globally asymptotically stable provided $(r + g_1 + g_2) > c$ in region Ω.

Proof. Take small perturbation in $E^*(S_1^*, I_1^*, S_2^*, I_2^*, D^*)$ as $v_1(t), v_2(t), v_3(t), v_4(t)$ and $v_5(t)$ respectively and putting $S_2 = S_2^* + v_3(t), I_2 = I_2^* + v_4(t)$ and $D = D^* + v_5(t)$ in set of equation of our model (2.1). After some manipulations the corresponding non linear system is given by

$$\frac{dv_1}{dt} = \left\{k\left(I_2^* + v_4\right) + r\right\} v_1 + \left(pb'\beta - kS_1^*\right) v_4 + hg_1 v_5,$$

$$\frac{dv_3}{dt} = \left\{k\left(I_1^* + v_2\right) + r\right\} v_3 + pb'(1-\beta) v_2 + \left\{pb'(1-\beta) - kS_2^*\right\} v_4 + (1-h) g_1 v_5,$$

$$\frac{dv_4}{dt} = k\left(I_1^* + v_2\right)v_3 + \left\{qb'(1-1) + kS_2^*\right\}v_2 + \left\{\delta_2\pi + qb'(1-1) - (r'+c)\right\}v_4,$$

$$\frac{dv_5}{dt} = cv_2 + cv_4 - \left(r + g_1 + g_2\right)v_5; \qquad \dots (1.1)$$

Let us take the following positive definite function of system (2.1) is

$$V = \left[v_1^2 + B_1v_2^2 + B_2v_3^2 + B_3v_4^2 + B_4v_5^2\right]/2. \qquad \dots (1.2)$$

Here B_i are arbitrary constant (i=1,2,3,4,5).

It time derivative is given by

$$\frac{dV}{dt} = v_1\frac{dv_1}{dt} + B_1v_2\frac{dv_2}{dt} + B_2v_3\frac{dv_3}{dt} + B_3v_4\frac{dv_4}{dt} + B_4v_5\frac{dv_5}{dt}. \qquad \dots (1.3)$$

We have positively invariant region Ω defined as

$$\Omega = \left\{\begin{matrix}(S_1, I_2, S_2, I_2, D) & 0 < S_{1m} \le S_1 \le P, & 0 < \bar{I}_1 \le I_1 \le \bar{P}, \\ & 0 < S_{2m} \le S_2 \le P, & 0 < \bar{I}_2 \le I_2 \le \bar{P}, \\ & 0 \le D \le \bar{D} & \end{matrix}\right\}.$$

Here $S_{1m}, \bar{I}_1, S_{2m}, \bar{I}_2$ are positive constants and

$$\bar{P} = \left[(\pi + b)\left(\bar{I}_1 + \bar{I}_2\right) + \left(\theta_1 + \theta_2\right)\right]/r_1,$$

where $r_1 = \min\left\{r, (r + g_1 + g_2)\right\}$

$\left(\bar{I}_1 + \bar{I}_2\right) = \left(S_1 + S_2\right)\left[1 + \left\{\pi + qb' - (r'+c)\right\}/k\right]$; and $D = c\left(\bar{I}_1 + \bar{I}_2\right)/(r + g_1 + g_2)$.

Here $\left(S_1 + S_2\right)$ is the number of susceptible males and females at initial level of study under the condition $(k + \pi + qb') > (r' + c)$.

Using the values of $\frac{dv_i}{dt}(i = 1,2,3,4,5)$ from (1) and using positively invariant region Ω given above in (1.3). we get

$$\frac{dv}{dt} = -\left(k\bar{I}_2 + r\right)v_1^2 - \left\{(r'+c) - (\delta_1\pi + qb'l)\right\}B_1v_2^2 - \left(k\bar{I}_1 + r\right)B_2v_3^2 - \left\{(r'+c) - \left(\delta_2\pi + qb'(1-l)\right)\right\}$$

$$B_3v_4^2 - (r + g_1 + g_2)B_4v_5^2 + \left\{k\bar{P}B_1 + pb'\beta\right\}v_1v_2 + \left\{pb'\beta - kS_1^*\right\}v_1v_4 + hg_1\, v_1v_5 +$$

$$\left\{\left(kS_1^* + qb'l\right)B_1\left(qb'(1-l) + kS_2^*\right)B_3\right\}v_2v_4 + pb'(1-\beta)v_2v_3$$

$$+\left\{k\bar{P}B_3 + \left(pb'(1-\beta) - kS_2^*\right)B_2\right\}v_3v_4 + (1-h)g_1B_2v_3v_5 + cB_4v_2v_5 + cB_4v_4v_5. \qquad \dots (1.4)$$

Applying inequality $ab \le \left(a^2+b^2\right)/2$ in (1.4) and after some manipulations, we derive

$$\frac{dv}{dt} \le -\left[B_{11}v_1^2 + B_{22}v_2^2 + B_{33}v_3^2 + B_{44}v_4^2 + B_{55}v_5^2\right]. \qquad \ldots (1.5)$$

Here

$$B_{11} = \left[(kI_1 + r) - \left\{2pb'\beta + k\bar{P}B_1 - kS_1^* + hg_1\right\}/2\right],$$

$$B_{22} = \left[\left\{(r'+c) - (\delta_1\pi + qb'l)\right\}B_1 + -\left\{pb'\beta + k\bar{P}B_1 + \left(qb'(1-l) + kS_2^*\right)B_3 + cB_4\right\}/2\right],$$

$$B_{33} = \left[\left(k\bar{I}_1 + r\right)B_2 - \left\{\left(2pb'(1-\beta) + hg_1 - kS_2^*\right)B_2 + k\bar{P}B_3\right\}/2\right],$$

$$B_{44} = \left[\left\{(r'+c) - \left(\delta_2\pi + qb'(1-l)\right)\right\}B_3 - \left\{pb'\beta - kS_1^* + \left(qb'l + kS_1^*\right)B_1\right.\right.$$

$$\left.\left. + \left(qb'(1-l) + kS_2^*\right)B_3 + \left(pb'(1-\beta) - kS_2^*\right)B_2 + k\bar{P}B_3 + cB_4\right\}/2\right],$$

$$B_{55} = \left[(r + g_1 + g_2)B_4 - \left\{hg_1 + (1-h)g_1B_2 + 2cB_4\right\}/2\right].$$

From (1.5) it can be shown that $\frac{dv}{dt}$ is negative definite in the region Ω under the following conditions:

$$\left(2I_2 + 2r + kS_1^*\right) > \left(2pb'\beta + kPB_1 + hg_1\right), \qquad \ldots (1.6)$$

$$\left\{(r'+c) - (\delta_1\pi + qb'l)\right\}B_1 > \frac{1}{2}\left\{pb'\beta + \left(k\bar{P} + qb'l + kS_1^*\right)B_1 + pb'(1-\beta)B_2 + \right.$$

$$\left.\left(qb'(1-l) + kS_1^*\right)B_3 + cB_4\right\}, \qquad \ldots (1.7)$$

$$\left(k\bar{I}_1 + r\right)B_2 > \frac{1}{2}\left\{\left(2pb'(1-\beta) + hg_1 - kS_2^*\right)B_2 + k\bar{P}B_3\right\}, \qquad \ldots (1.8)$$

$$\left\{(r'+c) - \left(\delta_2\pi + qb'(1-l)\right)\right\}B_3 > \frac{1}{2}\left\{pb'\beta - kS_1^* + \left(qb'l + kS_1^*\right)B_1 + \left\{pb'(1-\beta) - kS_2^*\right\}B_2\right.$$

$$\left.\left(qb'(1-l) + kS_2^* + k\bar{P}\right)B_3 + cB_4\right\}, \qquad \ldots (1.9)$$

$$(r + g_1 + g_2)B_4 > \frac{1}{2}\left\{hg_1 + (1-h)g_1B_2 + 2cB_4\right\}. \qquad \ldots (1.10)$$

The above conditions given by (1.6) to (1.10) may be further manipulated to get following simplified conditions:

From (1.7) choosing B_1

$$B_1 > \frac{\left[pb'\beta + pb'(1-\beta)B_2 + \left(qb'(1-l)kS_1^*\right)B_3 + cB_4\right]/2}{(r'+c) - \left(k\bar{P} + kS_1^* + 2\delta_1\pi + 3qb'l\right)/2},$$

the condition (1.7) reduces to

$$(r'+c)>\left(k\bar{P}+kS_1^*+2\delta_1\pi+3qb'l\right)/2. \quad \text{... (1.11)}$$

From (1.8) choosing as

$$B_2>\frac{\left[k\bar{P}B_3\right]/2}{\left[\left(k\bar{I}_1+r\right)-\left\{2pb'(1-\beta)+hg_1-kS_2^*\right\}/2\right]},$$

the condition (1.8) reduces to

$$\left[2\left(kI_1+r\right)+kS_2^*\right]>\left(2pb(1-\beta)+hg_1\right). \quad \text{... (1.12)}$$

From (1.9) choosing

$$B_3>\frac{\left[pb'\beta+\left(qb'l+kS_1^*\right)B_1+\left(pb'(1-\beta)-kS_2^*\right)B_2+cB_4\right]/2}{(r'+c)-\left(k\bar{P}+kS_2^*+2\delta_2\pi+3qb'(1-l)\right)/2},$$

along with $pb'(1-\beta)>kS_2^*$, the condition (1.9) reduces to

$$(r'+c)>\frac{1}{2}\left(k\bar{P}+kS_2^*+2\delta_2\pi+3qb'(1-l)\right). \quad \text{... (1.13)}$$

From the condition (1.10) choosing

$$B_4>\frac{\left[hg_1+(1-h)gB_2\right]/2}{\left(r+g_1+g_2\right)-c},$$

the condition (1.10) reduces to

$$(r+g_1+g_2)>c. \quad \text{.. (1.14)}$$

Thus, endemic steady state E^* is globally asymptotically stable in the region Ω under the condition (1.6), (1.11) to (1.14). Hence, it proves the **theorem**.

7. Numerical Analysis. To study the dynamical behavior of the model (2.1) numerically, the model is integrated by Runge-Kutta 4th method using the following set of parameter values:

$\theta_1=0.840, \theta_2=0.893, p=0.700, b'=0.801, \beta=0.200, g_1=0.200, k=0.578, r=0.400,$
$\delta_1=0.500, \delta_2=0.500, \pi=0.900, q=0.300, l=0.200, r'=1.000, c=0.600, g_2=0.136$

with initial conditions $S_1(0)=0.5, I_1(0)=0.3, S_2(0)=0.5, I_2(0)=0.3, D(0)=0.6$ the endemic steady state values are computed as shown graphically in Fig.3 (a-e). It is seen from the Fig. 3(a-e) that for any initial start:

(1) $S_1(0)=0.500, I_1(0)=0.300, S_2(0)=0.500, I_2(0)=0.300, D(0)=0.600$,

(2) $S_1(0)=0.745, I_1(0)=0.289, S_2(0)=0.845, I_2(0)=0.624, D(0)=0.414$,

(3) $S_1(0)=0.389, I_1(0)=0.654, S_2(0)=0.452, I_2(0)=0.498, D(0)=0.547$,

(4) $S_1(0)=0.687, I_1(0)=0.395, S_2(0)=0.712, I_2(0)=0.264, D(0)=0.870$,

the solution curve tend to steady state E^*. Hence we infer that the system (2.1) may be globally asymptotically stable about this endemic steady state E^*. The endemic steady statevalues of different variables are computed as follows:

$S_1^* = 1.05014931169431, I_1^* = 0.663578072430592,$

$S_2^* = 2.45490438471322, I_2^* = 1.11629241558393, D^* = 1.45098138082658$

We also compute numerically following variation matrix of system (2.1) around endemic steady state E^*

$$A_{E^*} = \begin{bmatrix} -1.0452 & 0.1121 & 0 & -0.4948 & 0.0400 \\ 0.6452 & -1.1019 & 0 & 0.6550 & 0 \\ 0 & -0.9704 & -0.7835 & 0.4486 & 01600 \\ 0 & 1.6112 & 0.3835 & -0.9578 & 0 \\ 0 & 0.6000 & 0 & 0.6000 & -0.7360 \end{bmatrix}.$$

The characteristic equation of variation matrix A_{E^*} is given as

$$\lambda^5 + 4.6265\lambda^4 + 7.2032\lambda^3 + 5.0926\lambda^2 + 1.5789\lambda + 0.1442 = 0$$

and the characteristic roots of variation matrix A_{E^*} are

$\lambda_1 = -2.3925182350361597882567301205 98,$

$\lambda_{2,3} = -0.71995847590301177926254602756 19$

$\quad \pm 0.3290294260549985883494208903371i,$

$\lambda_4 = -0.1497169755091599605142101142 6455,$

$\lambda_5 = -0.6423129797126565638772690342 5918.$

We also compute Routh-Hurwitz's array system (2.1) around endemic steady state $E*$ as

$$M_{E^*} = \begin{bmatrix} 1.000 & 7.2032 & 1.5789 \\ 4.6245 & 5.0926 & 0.1442 \\ 6.1020 & 1.5477 & 0 \\ 3.9196 & 0.1442 & 0 \\ 1.3233 & 0 & 0 \\ 0.1442 & 0 & 0 \end{bmatrix}.$$

Hence from characteristic roots of A_{E^*} and Routh-Hurwitz's array, we infer that the model (2.1) may be locally asymptotically stable around endemic steady state E^*.

In Fig. (4), the distribution of all sub-populations of system (2.1) with time is shown. It is found that susceptible (male and female) populations increase with the time due to the recruitment of susceptibles (male and female) and of new born susceptible offsprings. It is also seen that male susceptible population decreases rapidly in comparative to female susceptible population as population is not closed. Whereas, male and female infective populations, starts with small decrement and then increase rapidly with time due to initiation of Detection-program then it again decrease as all infective males and females have been effectively detected as HIV positive and got treatment. Its impact that the detected and treatmented population increases with time and then decrease slowly as they moved from the infection circulation due to either of their successful rehabilitation or dead from infection. Also some of them again stand as susceptible to infection circulation due to their improper rehabilitation. Finally the variations in all population are being stabilized due to endemicness of virus.

In Fig. 5(a-e), shows the variation of all sub-populations of system (2.1) with time at different level of horizontal transmission rates as $k=0.578$ to 0.800.It is found that all susceptible (male and female) populations decrease as transmission rate increase and all infective (male and female) populations along with deducted treatmented population firstly increase as transmission rate increase and then reach on its steady state position. The impact of recruitment (i.e. vertical transmission) of new born offsprings of infected parents directly into their corresponding classes, are explicitly shown in figure 6(a-e). It is seen in Fig. 6(a) and Fig.6(b) that at all increasing values of vertical transmission rate (b'), all susceptible populations firstly increase for some time, since population is not closed, then reach to its steady state position as earlier as transmission rate being increased. On the other hand in Fig. 6(c) and Fig. 6(d) as transmission rate increases, all infected and deducted - treatmented populations increase and reach to its steady state position as slowly as transmission rate increased with time. Thus if the birth of infected new born offsprings is controlled by medication, the overall infective population might be remained under control. That is, it would be helpful in reducing the density of infective populations which will transform into AIDS population.

The Fig. 7 (a-e) shows the effect of recruitment of infected immigrants to susceptible, infective and deducted–treatmented populations at different values of

$\pi = (0.912, 0.947, 0.984)$. It is seen that as recruitment rate of infected immigrants increased, all susceptible (male and female) populations reached earlier to its steady state position .This increase in rate also results that all infective and deducted-treatmented populations increase very quickly and then stabilized as slowly as rate increased. Hence, it may predict that the recruitment of infected immigrants is not only accelerating transmission efficiency but also increased viral density, which can create more complications to implement the best controlling strategies. Such satiation should be controlled because it may damage the controlling strategies more than vertical transmission.

The Fig. 8(a-e) predict the variation of all sub-population with time at different level (0.3 to o.7) of rate (partial failureness of D.T.R. program). It is seen that with the increase in the value of rate, the deducted and treatmented population decrease and reach to stabilize position earlier. Which results internal increase and late stabilize position to susceptible and infective populations. Hence, it would be fruitful to plane such controlling strategy that will focus not only on infection controlling but also on proper rehabilitation their education and socio-economic levels, so that failure chances of plane may be reduced.

8. Conclusion. In this paper, a deterministic mathematical model, have proposed and analyzed to understand the effect of infected immigrants on transmission dynamics of HIV infection in a heterosexual population, a controlling strategy named as Detection, Treatmentation and Rehabilitation Program (DTRP) is running. In this model, the growth of infection of infection is increased by sexual interaction between susceptible and infectives. The infection density is increased by new born offsprings of infected parents. The model also incorporated a partial failure rate of D.T.R. program to investigate the impact of socio-mythological, economical facts of targeted population, where D.T.R. program is running.

In the study of model, we have derived a threshold parameter R_0 and noted that, if $[\pi + qb' + k\theta/r] > (r' + c)$, the infection always remains in population. The model has a non-negative endemic steady state $E^*(S_1^*, I_1^*, S_2^*, I_2^*, D^*)$. It is found that, the endemic steady state E*, which exist only when? $R_0 > 1$, is always locally stable and globally asymptotically stable under the positively invariant region; obtained from model; with certain conditions.

It may speculate that an increase in the rate of vertical transmission leads to increase the populations of male infective and female infective; which would be turned into AIDS individuals. Thus the vertical spread of the infection should be controlled by effective promoting of condom use that will be kept the overall infective density under control. Also, it may be predicted that, by controlling the

recruitment of infected immigrants and by implementing an appropriate D.T.R. program according the situation of targeted population, the spread of infection can be reduced significantly.

It is found that an increase in the rate of recruitment of infective males and females leads an increase in the population of infectives as well as transmission rate of infection. It is shown that when viral load rapidly increase, controlling program has required to increase process accordingly, which may be reduced the effectiveness of D.T.R. program. Consequently the steady state values of infective males, infective females and detected-treatmented individuals can be maintained at desired levels. Hence the analysis of model, speculated that the HIV prevalence can be reduced and/or controlled by an effective strategy, which must include detection, treatment and rehabilitation process according targeted population, so that the progression to AIDS can be slowed stopped down.

Appendix

$$A_1 = [a-g+k-s+u];$$

$$A_2 = [k(a-g)-b(e-f)-ag-h(n+p)-l(q-r)+u(a-g+k-s)-s(a-g+k)];$$

$$A_3 = [(kl-h(v-w))(q-r)-(h(n+p)+l(q-r))(a-g+k)-u(ag+b(e-f)-k(a-g)+h(n+p)+ l(q-r)+s(a-g+k))-k(ag+b(e-f))+s(ag+b(e-f)-k(a-g))-(n+p)(gh+(b-c)(e-f))- dt(e-f)-mt(q-r)];$$

$$A_4 = [((kl-h(v-w))(q-r)-(n+p)(gh+(b-c)(e-f)))(a-g+k)-((v-w)(gh+(b-c)(e-f))+k(k l-h(v-w)))(q-r)-u((h(n+p)+l(q-r))(a-g+k)-(kl-h(v-w))(q-r)+k(ag+b(e-f)) s(ag+b(e-f)-k(a-g))+(n+p)(gh+(b-c)+(e-f)))-t(dg(e-f)-ad(e-f)+hm(q-r)) (dt(e-f)+mt(q-r))(a-g+k-s)(h(n+p)+l(q-r))(ag+b(e-f)-k(a-g))-(n+p)((e- f)(bh-a(b-c))+g(gh+(b-c)(e-f)))-t(ms(q-r)-km(q-r)+d(e-f)(n+p))+ ks(ag+b(e-f))];$$

$$A_5 = [dt(e-f)+mt(q-r))(ag+b(e-f)-k(a-g)+h(n-p)+l(q-r)+s(a-g+k))- t((n+p)(dg(e-f)-ad(e-f)+hm(q-r))+(q-r)(mk^2+lm(q-r)+d(e-f)(v- w))+s(ms(q-r)-km(q-r)+d(e-f)(n+p)))+u(((kl-h(v-w))(q-r)-(n+p)(gh+(b- c)(e-f)))(a-g+k)-v-w)(gh+(b-c)(e-f))+k(kl-h(v-w)))(q-r)+(h(n+p)+l(q- r))(ag+b(e-f)-k(a-g))-(n+p)((e-f)(bh-a(b-c))+g(gh+(b-c)(e-f)))+ks(ag+b(e- f))-(t(dg(e-f)-ad(e-f)+hm(q-r))+t(ms(q-r)-km(q-r)+d(e-f)(n+p)))(a-g+k- s)-t(g(dg(e-f)-ad(e-f)+hm(q-r))+(e-f)(da^2+bd(e-f)+m(b-c)(q-r))+h(ms(q- r)-km(q-r)+d(e-f)(n+p))];$$

Here

$$a = KI_2^* + r, b = pb'\beta, c = kS_1^*, d = g_1 h, e = kI_2^*, f = r, \; g = [\delta_1 \pi + qb'l],$$

$$h = [kS_1^* + qb'l], v = pb'(1-\beta), w = kS_2^*, k = KI_1^* + r, l = pb'(1-\beta); m = g_1(1-h)$$

$n = qb'(1-l), p = kS_2^*, s = [\delta_2 \pi + qb'(1-l)], t = c, q = kI_1^*, u = ($[illegible]$+ g_1 + g_2)$

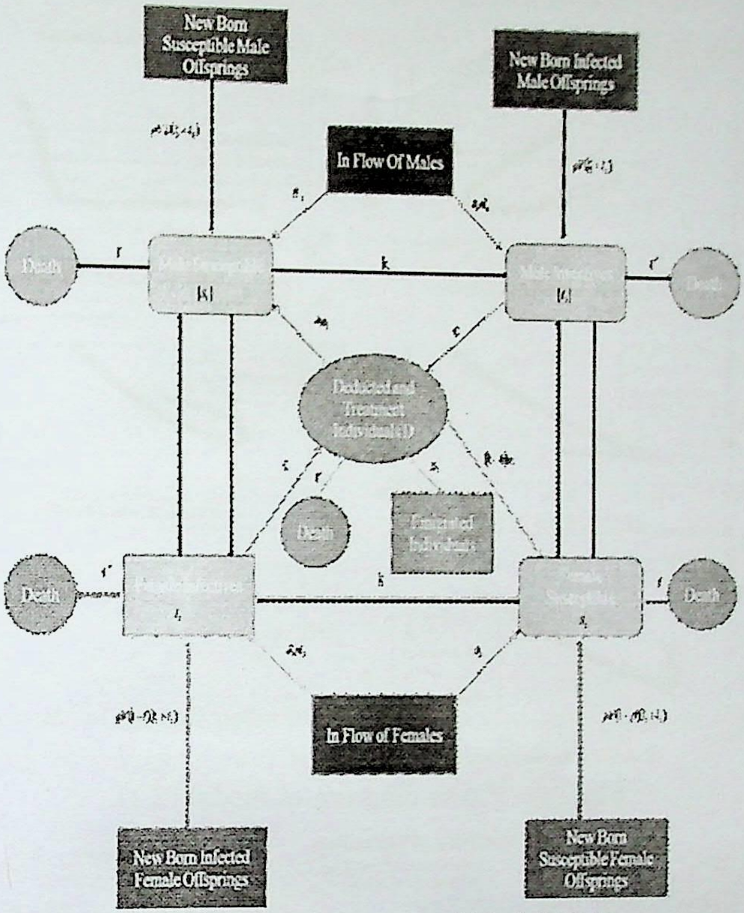


Fig.(1) : Flow Diagram of Model (2.1).

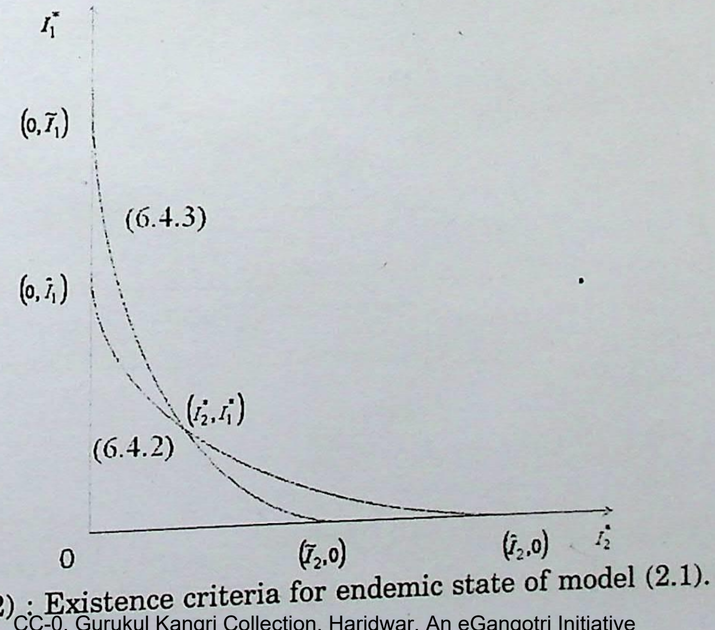


Fig. (2) : Existence criteria for endemic state of model (2.1).

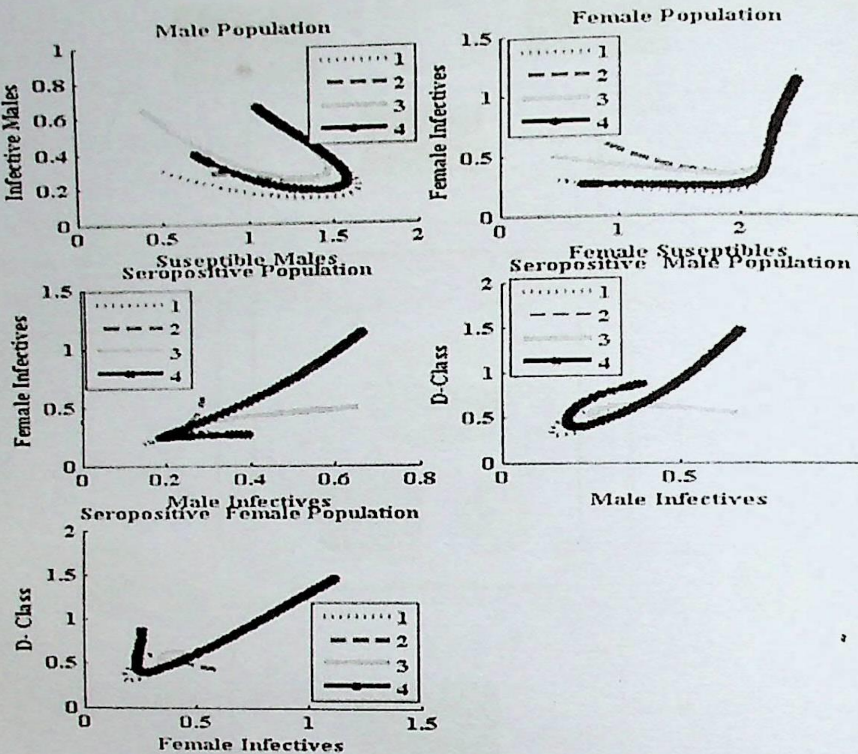


Fig. (3) : Phase diagram of model (2.1).

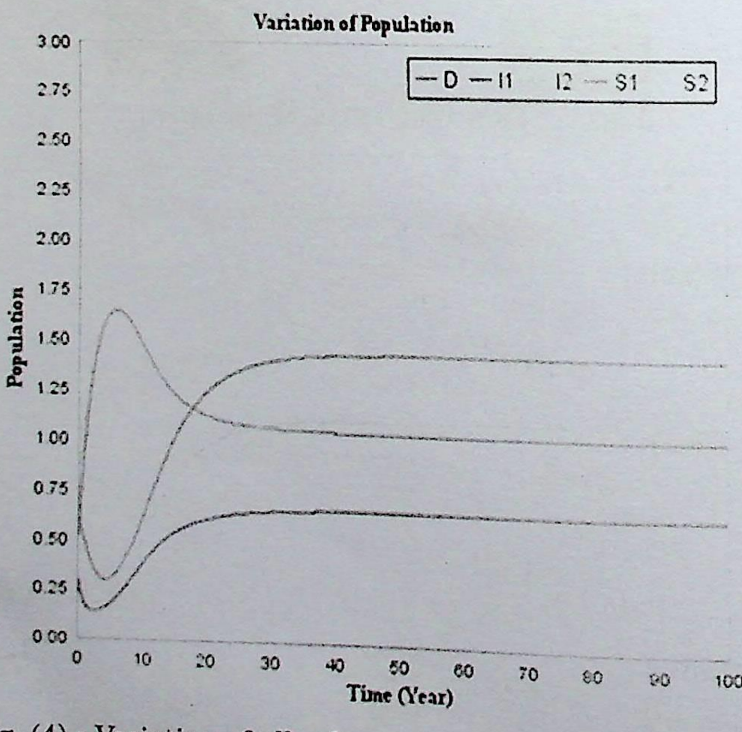


Fig. (4) : Variation of all sub population of model (2.1) with time.

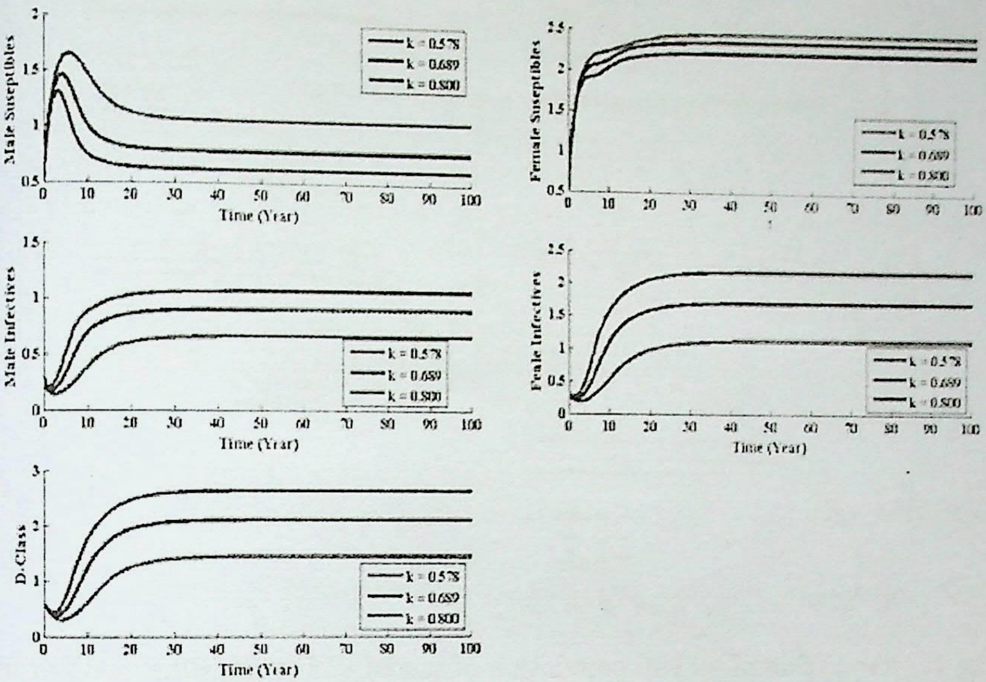


Fig. (5) : Variation of all sub population of model (2.1) with time at different values of transmission rate k as k =0.578, 0.689, 0.800.

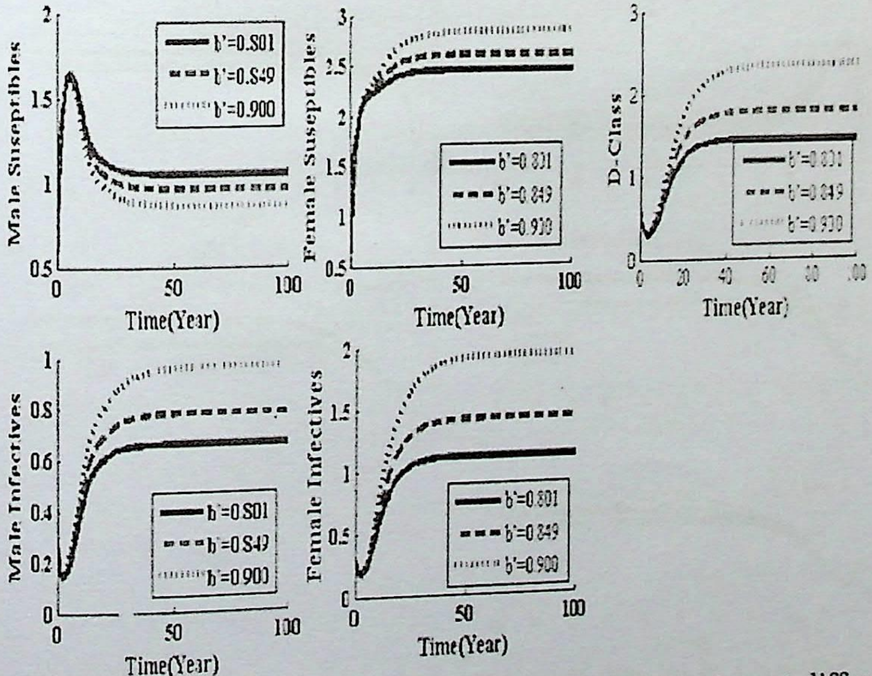


Fig. (6) : Variation of all sub population of model (2.1) with time at different values of birth rate b' of offsprings of infected parents as b'= 0.801, 0.849, 0.900.

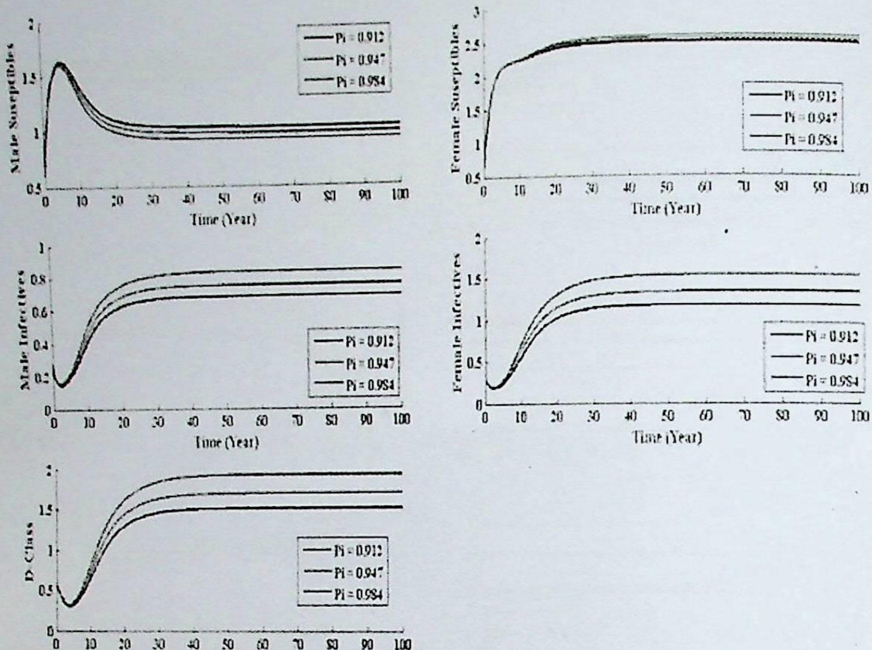


Fig. (7) : Variation of all sub population of model (2.1) with time at different values of recruitment of individuals ?, as ? = 0.912, 0.947, 0.984.

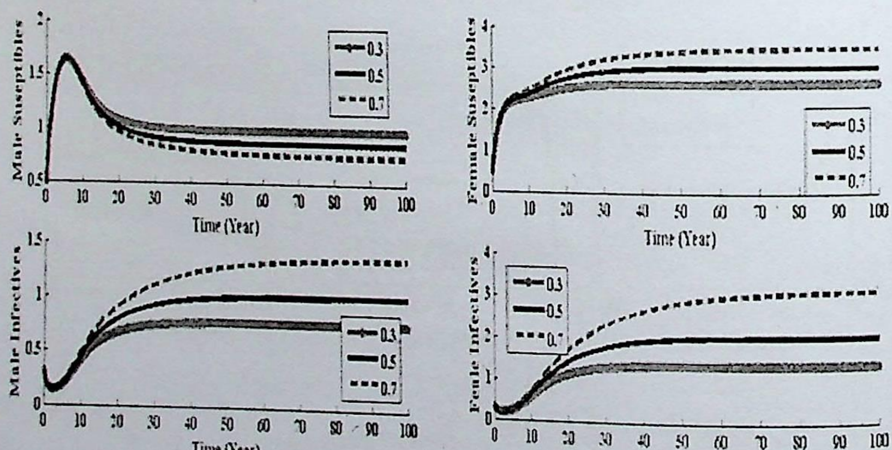


Fig. (8) : Variation of all Sub Population of Model (2.1) with time at different values of partial failure rate g1 of DTRP.

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A DEGENERATION FORMULA FOR EXTON'S QUADRUPLE HYPERGEOMETRIC FUNCTION $K_{12}(.)$ DUE TO A DIRICHLET TYPE INTEGRAL AND ITS APPLICATIONS

By

Hemant Kumar
Department of Mathematics, D.A-V. Postgraduate College, Kanpur-208001, Uttar Pradesh, India
E-mail: palhemant2007@rediffmail.com
M.A. Pathan
Centre for Mathematical Sciences, Pala Campus Arunapuram, P.O. Pala-686574, Kerala, India
E-mail:mapathan@gmail.com
and
Priyanka Yadav
Department of Mathematics, D.A-V. Postgraduate College, Kanpur-208001, Uttar Pradesh, India
E-mail: yadpriyanka9@yahoo.in

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ABSTRACT

In the present paper, we define a degeneration formula for a quadruple hypergeometric function $K_{12}(.)$ due to Exton [3,4] as introducing a Dirichlet type integral involving that quadruple hypergeometric function [3,4] and show that it is degenerated into bilinear product of double Appell functions $F_3(.)$ [cf. 1,11], and again, make its applications to derive fractional derivative representations, bilinear generating relations and transformation formulae. Finally, we present some particular cases, interesting applications, conclusions and discussions.

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Key Words and Phrases : Weighted Dirichlet type integral formula, Exton's quadruple hypergeometric function $K_{12}(.)$, fractional derivatives, bilinear generating relations, interesting applications, conclusions and discussions.

1. Introduction. From the normalized Dirichlet integral introduced due to Mathai and Haubold [6], particularly, in the four dimensional space $R^4 \subset C^5$, we define following Dirichlet integral density

(1.1) $I(\mu)=\int_{R^4} d_\mu(x)=$

$$\frac{1}{B(\mu)}\int_0^1\int_0^1\int_0^1\int_0^1 x_1^{\mu_1-1}x_2^{\mu_2-1}x_3^{\mu_3-1}x_4^{\mu_4-1}\left(1-x_1-x_2-x_3-x_4\right)^{\mu_5-1}dx_1dx_2dx_3dx_4$$

where, $x=(x_1,x_2,x_3,x_4)\in R^4$ such that $0\le x_1+x_2+x_3+x_4\le 1$, $\mu=(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5)$

$\in C^5$, $B(\mu)=\dfrac{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4)\Gamma(\mu_5)}{\Gamma(\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)}$ and $\mathrm{Re}(\mu_i)>0,\ \forall i=1,2,3,4,5.$

Exton [3,4] defined the following complete quadruple hypergeometric function among twenty one quadruple hypergeometric functions :

(1.2) $K_{12}(a,a,a,a,b_1,b_2,b_3,b_4;c_1,c_2,c_2;x,y,z,t)$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{m+n+p+q}(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(c_1)_{m+n}(c_2)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!},$$

where, the parameters $a,b_1,b_2,b_3,b_4,c_1,c_2$ are real or complex and x,y,z,t, are complex variables.

Now, for $\alpha>0,\beta>0,\gamma>0,\delta>0,\mathrm{Re}(\sigma_i)>\mathrm{Re}(-\mu_i),\mathrm{Re}(\mu_i)>0,\quad \forall i=1,2,3,4$ and

$\mathrm{Re}\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right)>0, a, b=(b_1,b_2,b_3,b_4), c=(c_1,c_2), \mu=(\mu_1,\mu_2,\mu_3,\mu_4), h_1,h_2,h_3$ and

$h_4\in C$ we define a function due to weighted Dirichlet type integral formula

(1.3) $F^{a,b,c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)=$

$$\frac{\Gamma(a)(\alpha)^{-(\mu_1+\sigma_1)}(\beta)^{-(\mu_2+\sigma_2)}(\gamma)^{-(\mu_3+\sigma_3)}(\delta)^{-(\mu_4+\sigma_4)}}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma\left(a-\sum_{i=1}^{4}(\mu i+\sigma_1)\right)}$$

$$\int_0^\alpha\int_0^\beta\int_0^\gamma\int_0^\delta\left(1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1}\right)^{a-\mu_1-\mu_2+\mu_3-\mu_4-1}\left(\frac{x}{1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1}}\right)^{\sigma_1}$$

$$\left(\frac{y}{1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1}}\right)^{\sigma_2}\left(\frac{z}{1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1}}\right)^{\sigma_3}\left(\frac{t}{1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1}}\right)^{\sigma_4}$$

$$x^{\mu_1-1}y^{\mu_2-1}z^{\mu_3-1}t^{\mu_4-1}K_{12}(a,a,a,a,b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;h_1x,h_2y,h_3z,h_4t)\,dxdydzdt$$

provided that $0 \le x\alpha^{-1} + y\beta^{-1} + z\gamma^{-1} + t\delta^{-1} \le 1$.

Particularly, when $b_1 = 0$, from (1.3), we have

$$(1.4) \quad F^{a(0,b_2,b_3,b_4),c,\mu,\sigma}\left(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta\right) =$$

$$= \frac{\Gamma(a)(\alpha)^{-(\mu_1+\sigma_1)}(\beta)^{-(\mu_2+\sigma_2)}(\gamma)^{-(\mu_3+\sigma_3)}(\delta)^{-(\mu_4+\sigma_4)}}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma\left(a - \sum_{i=1}^{4}(\mu_i+\sigma_i)\right)}$$

$$\int_0^\alpha\int_0^\beta\int_0^\gamma\int_0^\delta \left(1 - x\alpha^{-1} - y\beta^{-1} - z\gamma^{-1} - t\delta^{-1}\right)^{a-\mu_1-\mu_2-\mu_3-\mu_4-1}\left(\frac{x}{1 - x\alpha^{-1} - y\beta^{-1} - zy^{-1} - t\delta^{-1}}\right)^{\sigma_1}$$

$$\left(\frac{y}{1 - x\alpha^{-1} - y\beta^{-1} - zy^{-1} - t\delta^{-1}}\right)^{\sigma_2}\left(\frac{z}{1 - x\alpha^{-1} - y\beta^{-1} - zy^{-1} - t\delta^{-1}}\right)^{\sigma_3}\left(\frac{t}{1 - x\alpha^{-1} - y\beta^{-1} - zy^{-1} - t\delta^{-1}}\right)^{\sigma_4}$$

$$x^{\mu_1-1}y^{\mu_2-1}z^{\mu_3-1}t^{\mu_4-1}F_G\left(a,a,a,b_2,b_3,b_4;c_1,c_2,c_2;h_2y.h_3z,h_4t\right)dxdydzdt,$$

provided that $0 \le x\alpha^{-1} + y\beta^{-1} + z\gamma^{-1} + t\delta^{-1} \le 1$.

When, $b_1 = 0$, and $b_2 = 0$, (1.3) becomes

$$(1.5) \quad F^{a,(0,0,b_3,b_4),c,\mu,\sigma}\left(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta\right)$$

$$= \frac{\Gamma(a)(\alpha)^{-(\mu_1+\sigma_1)}(\beta)^{-(\mu_2+\sigma_2)}(\gamma)^{-(\mu_3+\sigma_3)}(\delta)^{-(\mu_4+\sigma_4)}}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma\left(a - \sum_{i=1}^{4}(\mu_i+\sigma_i)\right)}$$

$$\int_0^\alpha\int_0^\beta\int_0^\gamma\int_0^\delta \left(1 - x\alpha^{-1} - y\beta^{-1} - z\gamma^{-1} - t\delta^{-1}\right)^{a-\mu_1-\mu_2-\mu_3-\mu_4-1}\left(\frac{x}{1 - x\alpha^{-1} - y\beta^{-1} - zy^{-1} - t\delta^{-1}}\right)^{\sigma_1}$$

$$\left(\frac{y}{1 - x\alpha^{-1} - y\beta^{-1} - zy^{-1} - t\delta^{-1}}\right)^{\sigma_2}\left(\frac{z}{1 - x\alpha^{-1} - y\beta^{-1} - zy^{-1} - t\delta^{-1}}\right)^{\sigma_3}\left(\frac{t}{1 - x\alpha^{-1} - y\beta^{-1} - zy^{-1} - t\delta^{-1}}\right)^{\sigma_4}$$

$$x^{\mu_1-1}y^{\mu_2-1}z^{\mu_3-1}t^{\mu_4-1}F_1\left(a,b_3,b_4;c_2;h_3z,h_4t\right)dxdydzdt,$$

provided that $0 \le x\alpha^{-1} + y\beta^{-1} + z\gamma^{-1} + t\delta^{-1} \le 1$.

For, $b_1 = 0$, and $b_3 = 0$, (1.3) becomes

(1.6) $F^{a,(0,b_2,0,b_4),c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta) =$

$$= \frac{\Gamma(a)(\alpha)^{-(\mu_1+\sigma_1)}(\beta)^{-(\mu_2+\sigma_2)}(\gamma)^{-(\mu_3+\sigma_3)}(\delta)^{-(\mu_4+\sigma_4)}}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right)}$$

$$\int_0^\alpha\int_0^\beta\int_0^\gamma\int_0^\delta\left(1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1}\right)^{a-\mu_1-\mu_2+\mu_3-\mu_4-1}\left(\frac{x}{1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1}}\right)^{\sigma_1}$$

$$\left(\frac{y}{1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1}}\right)^{\sigma_2}\left(\frac{z}{1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1}}\right)^{\sigma_3}$$

$$\left(\frac{t}{1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1}}\right)^{\sigma_4} x^{\mu_1-1}y^{\mu_2-1}z^{\mu_3-1}t^{\mu_4-1}F_2(a,b_2,b_4;c_1,c_2;h_2y,h_4t)\,dxdydzdt,$$

provided that $0 \le x\alpha^{-1}+y\beta^{-1}+z\gamma^{-1}+t\delta^{-1} \le 1$.

For, $b_1=0, b_2=0$, and $b_3=0$, (1.3) becomes

(1.7) $F^{a,(0,0,0,b_4),c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta)$

$$= \frac{\Gamma(a)(\alpha)^{-(\mu_1+\sigma_1)}(\beta)^{-(\mu_2+\sigma_2)}(\gamma)^{-(\mu_3+\sigma_3)}(\delta)^{-(\mu_4+\sigma_4)}}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right)}$$

$$\int_0^\alpha\int_0^\beta\int_0^\gamma\int_0^\delta\left(1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1}\right)^{a-\mu_1-\mu_2-\mu_3-\mu_4-1}\left(\frac{x}{1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1}}\right)^{\sigma_1}$$

$$\left(\frac{y}{1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1}}\right)^{\sigma_2}\left(\frac{z}{1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1}}\right)^{\sigma_3}$$

$$\left(\frac{t}{1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1}}\right)^{\sigma_4} x^{\mu_1-1}y^{\mu_2-1}z^{\mu_3-1}t^{\mu_4-1}\,{}_2F_1(a,b_4;c_2;h_4t)\,dxdydzdt$$

provided that $0 \le x\alpha^{-1}+y\beta^{-1}+z\gamma^{-1}+t\delta^{-1} \le 1$.

Here, in our investigations, we show that when $\max\{|h_1\alpha|,|h_2\beta|\}<1$ and $\max\{|h_3\gamma|,|h_4\delta|\}<1$ the function $F^{a,b,c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta)$ defined in (1.3) degenerates into bilinear product of double Appell functions $F_3(.)$ (see, Appell and Kampé de Fériet [1] and Srivastava and Manocha [11,p.53]). Again, we make its applications to derive fractional derivative representations, bilinear generating

relations and transformation formulae. Finally, we present some particular cases, interesting applications, conclusions, and discussions.

2. Degeneration Formula and Fractional Derivative Representations. In this section, first we derive that the function $F^{a,b,c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta)$ defined in (1.3) degenerates into bilinear product of double Appell functions $F_3(.)$. (see, Appell and Kampé de Fériet [1] and Srivastava and Manocha [11,p.53]). Again, we make its applications to evaluate some fractional derivative representations.

Theorem A. When, $\max\{|h_1\alpha|, |h_2\beta|\} < 1$ and $\max\{|h_3\gamma|, |h_4\delta|\} < 1$, then, the function $F^{a,b,c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta)$ defined in (1.3) degenerates into bilinear product of double Appell functions $F_3(.)$, that is given by

(2.1) $F^{a,b,c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta) =$

$$F_3[\mu_1+\sigma_1, \mu_2+\sigma_2, b_1, b_2; c_1; h_1\alpha, h_2\beta] F_3[\mu_3+\sigma_3, \mu_4+\sigma_4, b_3, b_4; c_2; h_3\gamma, h_4\delta]$$

provided that all conditions given in (1.3) are satisfied.

Proof. In the right hand side of (1.3), for the given conditions $\alpha, \beta, \gamma, \delta > 0$, we set $x\alpha^{-1} = u_1, y\beta^{-1} = u_2, zy^{-1} = u_3,$ and $t\delta^{-1} = u_4$ and then define the quadruple hypergeometric function due to (1.2) and thus change the order of integration and summation, we find that

(2.2) $F^{a,b,c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta) =$

$$\frac{\Gamma(a)}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right)}$$

$$\sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(c_1)_{m+n}(c_2)_{p+q}} \frac{(h_1\alpha)^m}{m!} \frac{(h_2\beta)^n}{n!} \frac{(h_3\gamma)^p}{p!} \frac{(h_4\delta)^q}{q!} \int_0^1\int_0^1\int_0^1\int_0^1 (u_1)^{\mu_1+\sigma_1+m-1}$$

$$(u_2)^{\mu_2+\sigma_2+n-1}(u_3)^{\mu_3+\sigma_3+p-1}(u_4)^{\mu_4+\sigma_4+q-1}(1-u_1-u_2-u_3-u_4)^{\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right)-1} du_1du_2du_3du_4 \text{ , } \quad (2.2)$$

provided that $0 \le u_1+u_2+u_3+u_4 \le 1$.

Now, under the given conditions, $\mathrm{Re}(\sigma_i) > \mathrm{Re}(-\mu_i), \mathrm{Re}(\mu_i) > 0, \forall i = 1,2,3,4$ and $\mathrm{Re}\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right) > 0$, make an appeal to the formula (1.1) in right hand side

of (2.2), and again, in it define double Appell functions $F_3(.)$ [cf.1, 11], for the conditions, $\max\{|h_1\alpha|,|h_2\beta|\}<1$, and $\max\{|h_3\gamma|,|h_4\delta|\}<1$, we derive the result of (2.1).

Theorem B.

(i) If $\mathrm{Re}(c_1-\max\{b_2,(\mu_2+\sigma_2)\})>0, \mathrm{Re}(c_2-\max\{b_2,(\mu_4+\sigma_4)\})>0, |h_1\alpha|<1, |h_2\beta|<1$, $|h_3\gamma|<1$ and $|h_4\delta|<1$, then the fractional derivative representation of $F^{a,b,c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)$ defined in (1.3), is given by

(2.3) $F^{a,b,c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)=$

$$\frac{\Gamma(c_1-b_2)\Gamma(c_1-\mu_2-\sigma_2)\Gamma(c_2-b_4)\Gamma(c_2-\mu_4-\sigma_4)}{\Gamma(b_1)\Gamma(\mu_1+\sigma_1)\Gamma(b_3)\Gamma(\mu_3+\sigma_3)}(h_1)^{\mu_2+\sigma_2-c_1+1}(h_3)^{\mu_4+\sigma_4-c_2+1}D_{h_1}^{\mu_1+\sigma_1+\mu_2+\sigma_2-c_1}$$

$$D_{h_3}^{\mu_3+\sigma_3+\mu_4+\sigma_4-c_2}(h_1)^{\mu_1+\sigma_1+b_2-c_1}(h_3)^{\mu_3+\sigma_3+b_4-c_2}D_{h_1}^{b_1+b_2-c_1}D_{h_3}^{b_3+b_4-c_2}\Big\{(h_1)^{b_1-1}(h_3)^{b_3-1}(1-h_1\alpha)^{\mu_2+\sigma_2+b_2-c_1}$$

$$(1-h_3\gamma)^{\mu_4+\sigma_4+b_4-c_2}\,{}_2F_1\left[\begin{matrix}\mu_2+\sigma_2,b_2;\\c_1;\end{matrix}h_1\alpha+h_2\beta-h_1h_2\alpha\beta\right]{}_2F_1\left[\begin{matrix}\mu_4+\sigma_4,b_4;\\c_2;\end{matrix}h_3\gamma+h_4\delta-h_3h_4\gamma\delta\right]\Big\},$$

where $D_{h_1}=\dfrac{d}{dh_1}$, and $D_{h_3}=\dfrac{d}{dh_3}$.

(ii) If $\mathrm{Re}(c_1-\max\{b_2,(\mu_2+\sigma_2)\})>0, \mathrm{Re}(c_2-\max\{b_3,(\mu_3+\sigma_3)\})>0, |h_1\alpha|<1, |h_2\beta|<1$, $|h_3\gamma|<1$, and $|h_4\delta|<1$, then, the second fractional derivative representation of $F^{a,b,c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)$ defined in (1.3), is given by

(2.4) $F^{a,b,c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)=$

$$\frac{\Gamma(c_1-b_2)\Gamma(c_1-\mu_2-\sigma_2)\Gamma(c_2-b_3)\Gamma(c_2-\mu_3-\sigma_3)}{\Gamma(b_1)\Gamma(\mu_1+\sigma_1)\Gamma(b_4)\Gamma(\mu_4+\sigma_4)}(h_1)^{\mu_2+\sigma_2-c_1+1}(h_4)^{\mu_3+\sigma_3-c_2+1}D_{h_1}^{\mu_1+\sigma_1+\mu_2+\sigma_2-c_1}$$

$$D_{h_4}^{\mu_3+\sigma_3+\mu_4+\sigma_4-c_2}(h_1)^{\mu_1+\sigma_1+b_2-c_1}(h_4)^{\mu_4+\sigma_4+b_3-c_2}D_{h_1}^{b_1+b_2-c_1}D_{h_4}^{b_3+b_4-c_2}\Big\{(h_1)^{b_1-1}(h_4)^{b_4-1}(1-h_1\alpha)^{\mu_2+\sigma_2+b_2-c_1}$$

$$(1-h_4\delta)^{\mu_3+\sigma_3+b_3-c_2}\,{}_2F_1\left[\begin{matrix}\mu_2+\sigma_2,b_2;\\c_1;\end{matrix}h_1\alpha+h_2\beta-h_1h_2\alpha\beta\right]{}_2F_1\left[\begin{matrix}\mu_3+\sigma_3,b_3;\\c_2;\end{matrix}h_3\gamma+h_4\delta-h_3h_4\gamma\delta\right]\Big\},$$

where, $D_{h_1}=\dfrac{d}{dh_1}$, and $D_{h_4}=\dfrac{d}{dh_4}$.

Proof. Make an appeal to the fractional derivative formulae for double Appell function $F_3(.)$ due to Srivastava and Manocha [11, p. 306], we get

$$(2.5)\quad F_3[\alpha,\alpha',\beta,\beta';\gamma;\rho x,\sigma y]=\frac{\Gamma(\gamma-\alpha')\Gamma(\gamma-\beta')}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha'-\gamma+1}D_x^{\alpha+\alpha'-\gamma}x^{\alpha+\beta'-\gamma}D_x^{\beta+\beta'-\gamma}$$

$$\left\{x^{\beta-1}(1-\rho x)^{\alpha'+\beta'-\gamma}\;{}_2F_1\left[\begin{matrix}\alpha',\beta';\\ \gamma\end{matrix}\rho x+\sigma y-\rho\sigma xy\right]\right\},D_x=\frac{d}{dx}.$$

and

$$(2.6)\quad F_3[\alpha,\alpha',\beta,\beta';\gamma;\rho x,\sigma y]=\frac{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}{\Gamma(\alpha')\Gamma(\beta')}y^{\alpha-\gamma+1}D_y^{\alpha+\alpha'-\gamma}y^{\alpha'+\beta-\gamma}D_y^{\beta+\beta'-\gamma}$$

$$\left\{y^{\beta'-1}(1-\sigma y)^{\alpha+\beta-\gamma}\;{}_2F_1\left[\begin{matrix}\alpha,\beta;\\ \gamma;\end{matrix}\rho x+\sigma y-\rho\sigma xy\right]\right\},D_y=\frac{d}{dy}.$$

Then, use (2.5) in (2.1), we derive the result of (2.3) in the theorem B(i).
Again, use (2.5) and (2.6) in (2.1), we obtain the result of (2.4) in the theorem B(ii).

3. Generating Relations and Transformation Formulae. In this section, making some changes into the parameters of (2.1), we derive various results on generating relations and transformation formulae due to following theorems:

Theorem 1. If $|h_1\alpha|<r_1,|h_2\beta|<r_2,|T|<1,\frac{\beta|h_2|}{1+|T|^{-1}}<\left|\frac{\beta h_2 T}{T-1}\right|<r_2$ such that $r_1+r_2=r_1r_2$ and $|h_3\gamma|<r_3,|h_4\delta|<r_4,\frac{\delta|h_4|}{1+|T|^{-1}}<\left|\left(\frac{\delta h_4 T}{T-1}\right)\right|<r_4$ such that $r_3+r_4=r_3r_4$, then, for all conditions given in (1.3) and λ be any complex number, following bilinear generating relation holds

$$(3.1)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{a,(b_1,b_2-n,b_3,b_4-n),c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)T^n=$$

$$=(1-T)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu_2+\sigma_2)_n(\mu_4+\sigma_4)_n}{n!(c_1)_n(c_2)_n}\left\{\frac{h_2h_4\beta\delta T}{(1-T)^2}\right\}^n$$

$$F_S\left[\mu_1+\sigma_1,\mu_2+\sigma_2+n,\mu_2+\sigma_2+n,b_1,b_2,\lambda+n;c_1+n,c_1+n,c_1+n;h_1\alpha,h_2\beta,\frac{h_2\beta T}{T-1}\right]$$

$$F_S\left[\mu_3+\sigma_3,\mu_4+\sigma_4+n,\mu_4+\sigma_4+n,b_3,b_4,\lambda+n;c_2+n,c_2+n,c_2+n;h_3\gamma,h_4\delta,\frac{h_4\delta T}{T-1}\right],$$

where, $F_S[.]$ is the triple series defined due to Saran [8].

Proof. Replace b_2 by b_2-n, b_4 by b_4-n and multiply by $\frac{(\lambda)_n T^n}{n!}$ in both sides of (2.1), and then sum n from 0 to ∞ for $|T|<1$, we get

$$(3.2)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{a,(b_1,b_2-n,b_3,b_4-n),c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)T^n=$$

$$\sum_{p=0}^{\infty}\frac{(\mu_1+\sigma_1)_p(b_1)_p(h_1\alpha)^p}{(c_1)_p\,p!}\sum_{r=0}^{\infty}\frac{(\mu_3+\sigma_3)_r(b_3)_r(h_3\gamma)^r}{(c_2)_r\,r!}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}$$

$$\left\{\left(\sum_{p=0}^{\infty}\frac{(\mu_2+\sigma_2)_q(b_2-n)_q(h_2\beta)^q}{(c_1+p)_q\,q!}\right)\left(\sum_{s=0}^{\infty}\frac{(\mu_4+\sigma_4)_s(b_4-n)_s(h_4\delta)^s}{(c_2+r)_s\,s!}\right)\right\}T^n.$$

Now, in the right hand side of (3.2), use following formula due to Srivastava [10, p.71, eqn. (3.4)]

$$(3.3)\ \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(\sum_{l=0}^{\infty}\frac{(\rho-n)_l}{l!}\gamma_l x^l\right)\left(\sum_{m=0}^{\infty}\frac{(\sigma-n)_m}{m!}\delta_m y^m\right)T^n=(1-T)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left\{\frac{xyT}{(1-T)^2}\right\}^n$$

$$\sum_{l,m=0}^{\infty}\frac{(\rho)_l(\lambda+n)_m}{l!m!}\gamma_{n+l+m}x^l\left(\frac{xT}{T-1}\right)^m\sum_{u,v=0}^{\infty}\frac{(\sigma)_u(\lambda+n)_v}{u!v!}\delta_{n+u+v}y^u\left(\frac{yT}{T-1}\right)^v,|T|<1.$$

Then, make some series rearrangements and define the triple series $F_s[.]$ introduced due to Saran [8] (Also, see, Srivastava and Manocha [11,p.68], we obtain the result of (3.1).
In the similar manner, we have

Theorem 2. If $\max\left\{h_1\alpha,\frac{h_2\beta T}{T-1}\right\}<1,\max\left\{h_3\gamma,\frac{h_4\delta T}{T-1}\right\}<1$ and $|T|<1$, then, for all conditions given in (1.3) and λ be any complex, following bilinear generating relation holds

$$(3.4)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{a,(b_1,-n,b_3,-n),c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)T^n=$$

$$=(1-T)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu_2+\sigma_2)_n(\mu_4+\sigma_4)_n}{n!(c_1)_n(c_2)_n}\left\{\frac{h_2h_4\beta\delta T}{(1-T)^2}\right\}^n$$

$$F_3\left[\mu_1+\sigma_1,\mu_2+\sigma_2+n,b_1,\lambda+n;c_1+n;h_1\alpha,\frac{h_2\beta T}{T-1}\right]$$

$$F_3\left[\mu_3+\sigma_3;\mu_4+\sigma_4+n,b_2,\lambda+n;c_2+n;h_3\gamma,\frac{h_4\delta T}{T-1}\right].$$

Theorem 3. If $|h_1\alpha|<\infty,|h_2\beta|<\infty,|T|<1,$ and $|h_3\gamma|<\infty,|h_4\delta|<\infty$, then, for all conditions given in (1.3) and λ be any complex, following bilinear generating relation holds

$$\text{(3.5)}\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{a,(b_1,b_2-n,b_3,b_4+n),c,\mu,\sigma}(h_1\alpha;h_2\beta,h_3\gamma,h_4\delta)T^n=$$

$$(1-T)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu_2+\sigma_2)_n(\mu_4+\sigma_4)_n}{n!(c_1)_n(c_2)_n}\left\{-\frac{h_2h_4\beta\delta T}{(1-T)^2}\right\}^n$$

$$F^{(3)}\left[\begin{matrix}(-)::(-);(\mu_2+\sigma_2+n)_1;(-):(\mu_1+\sigma_1,b_1)_2;(b_2)_1;(\lambda+n)_1:\\(c_1+n)_1::(-);(-);(-):(-);(-);(-):\end{matrix}h_1\alpha,h_2\beta,\frac{h_2\beta T}{T-1}\right]$$

$$F^{(3)}\left[\begin{matrix}(-)::(-);(\mu_4+\sigma_4+n)_1;(-):(\mu_3+\sigma_3,b_3)_2;(b_4-\lambda)_1;(\lambda+n)_1:\\(c_2+n)_1::(-);(-);(-):(-);(-);(-):\end{matrix}h_3\gamma,h_4\delta,\frac{h_4\delta}{1-T}\right],$$

where, $F^{(3)}[...]$ is the general triple hypergeometric function due to Srivastava [9].

Theorem 4. If $|h_1\alpha|<\infty,|h_2\beta|<\infty,|T|<1,$ and $|h_3\gamma|<\infty,|h_4\delta|<\infty$, such that then, for all conditions given in (1.3) and λ be any complex number, following bilinear generating relation holds

$$\text{(3.6)}\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{a,(b_1,b_2+n,b_3,b_4+n),c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)T^n=$$

$$=(1-T)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu_2+\sigma_2)_n(\mu_4+\sigma_4)_n}{n!(c_1)_n(c_2)_n}\left\{\frac{h_2h_4\beta\delta T}{(1-T)^2}\right\}^n$$

$$F^{(3)}\left[\begin{matrix}(-)::(-);(\mu_2+\sigma_2+n)_1;(-):(\mu_1+\sigma_1,b_1)_2;(b_2-\lambda)_1;(\lambda+n)_1:\\(c_1+n)_1::(-);(-);(-):(-);(-);(-):\end{matrix}h_1\alpha,h_2\beta,\frac{h_2\beta}{1-T}\right]$$

$$F^{(3)}\left[\begin{matrix}(-)::(-);(\mu_4+\sigma_4+n)_1;(-):(\mu_3+\sigma_3,b_3)_2;(b_4-\lambda)_1;(\lambda+n)_1:\\(c_2+n)_1::(-);(-);(-):(-);(-);(-):\end{matrix}h_3\gamma,h_4\delta,\frac{h_4\delta}{1-T}\right],$$

where, $F^{(3)}[...]$ is the general triple hypergeometric function due to Srivastava [9].

Theorem 5. If $\max\{|h_1\alpha|,|h_2\beta|\}<1, |T|<1, \left|\frac{h_3\gamma}{T-1}\right|<\infty, |h_4\delta|<1$, then, for all conditions given in (1.3) and λ be any complex number, following bilinear generating relation holds

$$(3.7)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{a,(b_1,b_2-n,\lambda+n,b_4),c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)T^n =$$

$$=(1-h_4\delta)^{-(\mu_4+\sigma_4)}(1-T)^{-\lambda}\sum_{n=0}^{\infty}\frac{(h_1\alpha)^n}{n!(c_1)_n}\sum_{m=0}^{\infty}\frac{(-n)_m(\mu_1+\sigma_1)_{n-m}(b_1)_{n-m}(\lambda)_m(\mu_2+\sigma_4)_m}{m!}\left\{\frac{h_2\beta T}{h_1\alpha(1-T)}\right\}^m$$

$$F^{(3)}\left[\begin{matrix}(-)::(-);(c_2-b_4)_1;(-):(\mu_2+\sigma_2+m,b_2)_2;(\lambda+m,\mu_3+\sigma_3)_2;(\mu_4+\sigma_4)_1: \\ (-)::(-);(c_2)_1;(-):(c_1+n)_1;(c_2-b_4)_1;(-):\end{matrix} h_2\beta,\frac{h_3\gamma}{1-T},\frac{h_4\delta}{h_4\delta-1}\right]$$

Proof. In (2.1), replace b_2 by b_2-n and b_3 by $\lambda+n$ and then multiply by $\frac{(\lambda)_n T^n}{n!}$ in both sides to it and thus sum n fro 0 to ∞ for $|T|<1$, we may write

$$(3.8)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{a,(b_1,b_2-n,\lambda+n,b_4),c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)T^n =$$

$$=(1-h_4\delta)^{-(\mu_4+\sigma_4)}\sum_{m=0}^{\infty}\frac{(\mu_1+\sigma_1)_m(b_1)_m(h_1\alpha)^m}{m!(c_1)_m}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2F_1\left[\begin{matrix}b_2-n,\mu_2+\sigma_2; \\ c_1+m;\end{matrix} h_2\beta\right]$$

$$F^{1:2;1}_{1:1;0}\left[\begin{matrix}c_2-b_4:\lambda+n,\mu_3+\sigma_3;\mu_4+\sigma_4; \\ c_2:c_2-b_4;-;\end{matrix} h_3\gamma,\frac{h_4\delta}{h_4\delta-1}\right]T^n.$$

Now, for $|T|<1$, in the right hand side of (3.8), making an appeal to the formula due to Srivastava and Manocha [11,p.155]

$$(3.9)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_{l+1}F_m\left[\begin{matrix}\rho-n,(g_l); \\ (h_m);\end{matrix} x\right]F^{u:p+1;r}_{v:q;s}\left[\begin{matrix}(e_u):\lambda+n,(a_p);(c_r); \\ (f_v):(b_q);(d_s);\end{matrix} y,z\right]T^n =$$

$$(1-T)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n\prod_{j=1}^{l}(g_j)_n}{n!\prod_{j=1}^{m}(h_j)_n}\left(\frac{xT}{T-1}\right)^n$$

$$F^{(3)}\left[\begin{matrix}(-)::(-);((e_u))_u;(-):(\rho,(g_l)+n)_{l+1};(\lambda+n,(a_p))_{p+1}:((c_r))_r; \\ (-)::(-);((f_v))_v;(-):((h_m)+n)_m;((b_q))_q;((d_s))_s;\end{matrix} x,\frac{y}{1-T},z\right],$$

where, $F^{u:p;r}_{v;q,r}[.,.]$ is the two variables Kampé de Fériet function due to Srivastava and Panda [12] and $F^{(3)}[.,.,.]$ is the general triple hypergeometric function due to Srivastava [9]. Then using series rearrangement techniques, we obtain the result (3.7).

Theorem 6. If $\max\{|h_1\alpha|,|h_2\beta|\}<1, |T|<1, \left|\frac{h_3\gamma}{T-1}\right|<\infty, |h_4\delta|<1$, then, for all conditions given in (1.3) and λ be any complex number, following bilinear generating relation holds

$$(3.10)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{a,(b_1,b_2-n,\lambda+n,b_4),c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)T^n =$$

$$(1-h_4\delta)^{-(b_4)}(1-T)^{-\lambda}\sum_{n=0}^{\infty}\frac{(h_1\alpha)^n}{n!(c_1)_n}\sum_{m=0}^{n}\frac{(-n)_m(\mu_1+\sigma_1)_{n-m}(b_1)_{n-m}(\lambda)_m(\mu_2+\sigma_2)_m}{m!}\left\{\frac{h_2\beta T}{h_1\alpha(1-T)}\right\}^m$$

$$F^{(e)}\left[\begin{matrix}(-)::(-);(c_2-\mu_4-\sigma_4)_1;(-):(\mu_2+\sigma_2+m,b_2)_2;(\lambda+m,\mu_3+\sigma_3)_2;(b_4)_1: \\ (-)::(-);(c_2)_1;(-):(c_1+n)_1;(c_2-\mu_4-\sigma_4)_1;(-):\end{matrix}\right.$$

$$\left. h_2\beta,\frac{h_3\gamma}{1-T},\frac{h_4\delta}{h_4\delta-1}\right].$$

Theorem 7. If $\max\{|h_1\alpha|,|h_2\beta|\}<1, |T|<1, \max\{|h_3\gamma|,|h_4\delta|\}<1$, then, for all conditions given in (1.3) and λ be any complex number, following transformation formula holds

$$(3.11)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{a,(b_1,\lambda+n,b_3,\lambda+n),c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)T^n$$

$$=(1-T)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu_2+\sigma_2)_n(\mu_4+\sigma_4)_n}{n!(c_1)_n(c_2)_n}\left\{\frac{h_2h_4\beta\delta T}{(1-T)^2}\right\}^n$$

$$F^{a,(b_1,\lambda+n,b_3,\lambda+n),(c_1+n,c_2+n),(\mu_1,\mu_2+n,\mu_3,\mu_4+n),\sigma}\left(h_1\alpha,\frac{h_2\beta}{1-T},h_3\gamma,\frac{h_4\delta}{1-T}\right).$$

Theorem 8. If $\max\{|h_1\alpha|,|h_2\beta|\}<1, |T|<1, \max\{|h_3\gamma|,|h_4\delta|\}<1$, then, for all conditions given in (1.3) and λ be any complex number, following transformation formula holds

$$(3.12)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{a,(b_1,-n,b_3,\lambda+n),c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)T^n$$

$$= (1-T)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n (\mu_2+\sigma_2)_n (\mu_4+\sigma_4)_n}{n!(c_1)_n (c_2)_n} \left\{ -\frac{h_2 h_4 \beta\delta T}{(1-T)^2} \right\}^n$$

$$F^{a,(b_1,\lambda+n,b_3\lambda+n),(c_1+n,c_2+n),(\mu_1,\mu_2+n,\mu_3,\mu_4+n),\sigma}\left(h_1\alpha, \frac{h_2\beta T}{T-1}, h_3\gamma, \frac{h_4\delta}{1-T} \right).$$

4. Special Cases and Interesting Applications. Setting $h_2\beta = -h_1\alpha$, $h_4\delta = -h_3\gamma, b_1 = \mu_1 + \sigma_1, b_2 = \mu_2 + \sigma_2, b_3 = \mu_3 + \sigma_3$, and $b_4 = \mu_4 + \sigma_4$, in (2.1), we find that

(4.1) $$F^{a,(\mu_1+\sigma_1,\mu_2+\sigma_2,\mu_3+\sigma_3,\mu_4+\sigma_4),c,\mu,\sigma}(h_1\alpha, -h_1\alpha, h_3\gamma, -h_3\gamma) =$$

$$= {}_4F_3\left[\begin{matrix} \mu_1+\sigma_1, \mu_2+\sigma_2, (\mu_1+\sigma_1+\mu_2+\sigma_2)/2, (\mu_1+\sigma_1+\mu_2+\sigma_2+1)/2; \\ \mu_1+\sigma_1+\mu_2+\sigma_2, (c_1)/2, (c_1)/2+1/2; \end{matrix} (h_1\alpha)^2 \right]$$

$${}_4F_3\left[\begin{matrix} \mu_3+\sigma_3, \mu_4+\sigma_4, (\mu_3+\sigma_3+\mu_4+\sigma_4)/2, (\mu_3+\sigma_3+\mu_4+\sigma_4+1)/2; \\ \mu_3+\sigma_3+\mu_4+\sigma_4, (c_2)/2, (c_2)/2+1/2; \end{matrix} (h_3\gamma)^2 \right].$$

Again, setting $b_1 = 1, b_2 = 1, b_3 = 1$, and $b_4 = 1, c_1 = \mu_1 + \sigma_1 + \mu_2 + \sigma_2$ and $c_2 = \mu_3 + \sigma_3 + \mu_4 + \sigma_4$ in (2.1), we find that

(4.2) $$F^{a,(1,1,1,1),(\mu_1+\sigma_1+\mu_2+\sigma_2,\mu_3+\sigma_3+\mu_4+\sigma_4),\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta) = (h_1\alpha + h_2\beta - h_1\alpha h_2\beta)^{-1}$$

$$(h_3\gamma + h_4\delta h_3\gamma h_4\delta)^{-1} \left\{ h_1\alpha\, {}_2F_1\left[\begin{matrix} \mu_1+\sigma_1, 1; \\ \mu_1+\sigma_1+\mu_2+\sigma_2; \end{matrix} h_1\alpha \right] + h_2\beta\, {}_2F_1\left[\begin{matrix} \mu_2+\sigma_2, 1; \\ \mu_1+\sigma_1+\mu_2+\sigma_2; \end{matrix} h_2\beta \right] \right\}$$

$$\left\{ h_3\gamma\, {}_2F_1\left[\begin{matrix} \mu_3+\sigma_3, 1; \\ \mu_3+\sigma_3+\mu_4+\sigma_4; \end{matrix} h_3\gamma \right] + h_4\delta\, {}_2F_1\left[\begin{matrix} \mu_4+\sigma_4, 1; \\ \mu_3+\sigma_3+\mu_4+\sigma_4; \end{matrix} h_4\delta \right] \right\}.$$

Setting $c_1 = 2b_2, c_2 = 2b_3, \mu_2 = b_2 - 1/2 - \sigma_2, \mu_3 = b_3 - 1/2 - \sigma_3$ in (2.6), then, we find

(4.3) $$F^{a,b,(2b_2,2b_3),(\mu_1,b_2-1/2-\sigma_2,b_3-1/2-\sigma_3,\mu_4),\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta) =$$

$$= (4)^{b_2+b_3-1} \frac{\Gamma(b_2+1/2)\Gamma(b_2)\Gamma(b_3+1/2)\Gamma(b_3)}{\Gamma(\mu_1+\sigma_1)\Gamma(b_1)\Gamma(\mu_4+\sigma_4)\Gamma(b_4)} (h_1)^{1/2-b_2} (h_4)^{1/2-b_3} D_{h_1}^{\mu_1+\sigma_1-b_2+1/2} D_{h_4}^{\mu_4+\sigma_4-b_3-1/2}$$

$$\left(1+\sqrt{(1-h_1\alpha)(1-h_2\beta)}\right)^{1-2b_2} \left(1+\sqrt{(1-h_3\gamma)(1-h_4\delta)}\right)^{1-2b_3} \Big\},$$

provided that $\mathrm{Re}(b_2) > -1/2, \mathrm{Re}(b_3) > -1/2, |h_1\alpha| < 1, |h_2\beta| < 1, |h_3\gamma| < 1$, and $|h_4\delta| < 1$. Now, from (4.3), we obtain an interesting transformation formula

(4.4) $$F^{a,b,(2b_2,2b_3),\left(\mu_1,b_2-\frac{1}{2}-\sigma_2,b_3-\frac{1}{2}-\alpha_3,\mu_4\right),\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta) =$$

$$= (4)^{b_2+b_3-1} M^{b_2} \left\{ {}_3F_2\left[\begin{matrix} b_1, 1/2-\xi/2, \mu_1+\sigma_1; \\ b_2, b_2+1/2; \end{matrix} h_1\alpha \right] \right\} (h_2\beta)$$

$$M^{b_3}\left\{ {}_3F_2\left[\begin{matrix} b_4, 1/2-\eta/2, \mu_4+\sigma_4; \\ b_3, b_3+1/2; \end{matrix} h_4\delta\right]\right\}(h_3\gamma). \tag{4.4}$$

provided that $\mathrm{Re}(b_2) > -1/2, \mathrm{Re}(b_3) > -1/2, |h_1\alpha| < 1, |h_4\delta| < 1, \left|\arg\left(\sqrt{(1-h_2\beta)}\right)\right| < \pi$, and $\left|\arg\left(\sqrt{(1-h_3\gamma)}\right)\right| < \pi$.

Here, the Mellin-Barnes transformation operator is defined as

(4.5) $$M^{b_2}\{\varphi(\xi)\}(h_2\beta) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(2b_2-1+\xi)\Gamma(-\xi)(1-h_2\beta)^{\xi/2}}{\Gamma(2b_2-1)}\{\varphi(\xi)\}d\xi.$$

Further, we evaluate second interesting result in the form

(4.6) $$F^{a,b,(2b_2,2b_3),(u_1,b_2-1/2-\sigma_2,b_3-1/2-\sigma_3,\mu_4),\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta) =$$

$$= \frac{(4)^{b_2+b_3-1}}{\Gamma(2b_2-1)\Gamma(2b_3-1)} \sum_{n,m=0}^{\infty} \frac{(b_1)_n(\mu_1+\sigma_1)_n(b_4)_m(\mu_4+\sigma_4)_m(h_1\alpha)^n(h_4\delta)^m}{(b_2+1/2)_n(b_3+1/2)_m\, n!m!}$$

$$H_{2,2}^{2,1}\left[\sqrt{(1-h_2\beta)}\left|\begin{matrix}(2-2b_2,1),(1/2,1/2)\\(0,1),(1/2+m,1/2)\end{matrix}\right.\right] H_{2,2}^{2,1}\left[\sqrt{(1-h_3\gamma)}\left|\begin{matrix}(2-2b_3,1),(1/2,1/2)\\(0,1),(1/2+m,1/2)\end{matrix}\right.\right],$$

provided that $\mathrm{Re}(b_2) > -1/2, \mathrm{Re}(b_3) > -1/2, |h_1\alpha| < 1, |h_4\delta| < 1, \left|\arg\left(\sqrt{(1-h_2\beta)}\right)\right| < \pi$, and $\left|\arg\left(\sqrt{(1-h_3\gamma)}\right)\right| < \pi$.

Here, the Fox's H-function [5] (Also see, Srivastava and Manocha [11,p.49]) is given by

(4.7) $$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_p,A_p)\\(\beta_1,B_1),\ldots,(\beta_q,B_q)\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{m}\Gamma(\beta_j-B_j\xi)\prod_{j=1}^{n}\Gamma(1-\alpha_j+A_j\xi)z^{\xi}}{\prod_{j=m+1}^{q}\Gamma(1-\beta_j+B_j\xi)\prod_{j=n+1}^{p}\Gamma(\alpha_j-A_j\xi)}d\xi.$$

The integral in (4.7) converges absolutely if

(4.8) $$\Omega = \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j + \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j > 0$$

and analytic in the sector

(4.9) $|\arg(z)| < 1/2\Omega\pi$.

S e $b_1=0$ and $b_3=0$ in (3.1), we get a result equivalent to the result due to Srivastava and Manocha [11, p.148, eqn. (39), for $p=q=r=s=1$]

(4.10) $$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{a,(0,b_2-n,0,b_4-n),c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta)T^n =$$

$$= (1-T)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n (\mu_2+\sigma_2)_n (\mu_4+\sigma_4)_n}{n!(c_1)_n (c_2)_n} \left\{ \frac{h_2 h_4 \beta\delta T}{(1-T)^2} \right\}^n$$

$$F_1\left[\mu_2+\sigma_2+n, b_2, \lambda+n; c_1+n; h_2\beta, \frac{h_2\beta T}{T-1} \right] \cdot$$

$$F_1\left[\mu_4+\sigma_4+n, b_4, \lambda+n; c_2+n; h_4\delta, \frac{h_4\delta T}{T-1} \right], |T|<1.$$

When, $b_1=0$, by direct computation of integral of (1.4) of from (2.1), we find that

(4.11) $F^{a,(0,b_2,b_3,b_4),c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta) =$

$$ {}_2F_1\left[\begin{matrix} \mu_2+\sigma_2, b_2; \\ c_1; \end{matrix} h_2\beta \right] F_3\left[\mu_3+\sigma_3, \mu_4+\sigma_4, b_3, b_4; c_2; h_3\gamma, h_4\delta \right].$$

When, $b_1=0$, and $b_2=0$, by direct computation of integral of (1.5) or from (2.1), we find that

(4.12) $F^{a,(0,0,b_3,b_4),c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta) = F_3\left[\mu_3+\sigma_3, \mu_4+\sigma_4, b_3, b_4; c_2; h_3\gamma, h_4\delta \right].$

When, $b_1=0$, and $b_3=0$, by direct computation of integral of (1.6) or from (2.1), we find that

(4.13) $F^{a,(0,b_2,0,b_4),c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta) = {}_2F_1\left[\begin{matrix} \mu_2+\sigma_2, b_2; \\ c_1; \end{matrix} h_2\beta \right] {}_2F_1\left[\begin{matrix} \mu_4+\sigma_4, b_4; \\ c_2; \end{matrix} h_4\delta \right].$

When, $b_1=0, b_2=0$, and $b_3=0$, by direct computation of integral of (1.7) or from (2.1), we find

(4.14) $F^{a,(0,0,0,b_4),c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta) = {}_2F_1\left[\begin{matrix} \mu_4+\sigma_4, b_4; \\ c_2; \end{matrix} h_4\delta \right].$

Setting $b_1=0$, $b_2=0$, $b_3=0$, $b_4=0$, $T=-S$ and $\lambda=-\eta$ in both sides of (3.1) and then, in left hand side of that on making use of (2.1), particularly, we find the generating relation due to Meixner [7] (Also, see Erdélyi [2, eqn. (8), p. 84])

(4.15) $\sum_{n=0}^{\infty} \binom{\eta}{n} {}_2F_1\left[\begin{matrix} \mu_2+\sigma_2, -n; \\ c_1; \end{matrix} h_2\beta \right] {}_2F_1\left[\begin{matrix} \mu_4+\sigma_4, -n; \\ c_2; \end{matrix} h_4\delta \right] S^n =$

$$= (1+S)^{\eta} \sum_{n=0}^{\infty} \binom{\eta}{n} \frac{(\mu_2+\sigma_2)_n (\mu_4+\sigma_4)_n}{n!(c_1)_n (c_2)_n} \left\{ \frac{h_2 h_4 \beta\delta S}{(1+S)^2} \right\}^n {}_2F_1\left[\begin{matrix} \mu_2+\sigma_2+n, n-\eta; \\ c_1+n; \end{matrix} \frac{h_2\beta S}{1+S} \right]$$

$$ {}_2F_1\left[\begin{matrix} \mu_4+\sigma_4+n, n-\eta; \\ c_2+n; \end{matrix} \frac{h_4\delta S}{1+S} \right].$$

Again, setting $b_1=0, b_3=0, b_2=-\eta, b_4=-\eta, T=-S$ and $\lambda=-\eta$ in both sides of (3.6) and then, in left hand side of that on making use of (2.1), particularly, we find the generating relation due to Meixner [7] (Also, see Erdélyi [2, eqn. (10), p.84])

$$(4.16)\quad \sum_{n=0}^{\infty}\binom{\eta}{n}\,{}_2F_1\left[\begin{matrix}\mu_2+\sigma_2, n-\eta;\\ c_1;\end{matrix}\, h_2\beta\right]{}_2F_1\left[\begin{matrix}\mu_4+\sigma_4, n-\eta;\\ c_2;\end{matrix}\, h_4\delta\right]S^n =$$

$$=(1+S)^{\eta}\sum_{n=0}^{\infty}\binom{\eta}{n}\frac{(\mu_2+\sigma_2)_n(\mu_4+\sigma_4)_n}{n!(c_1)_n(c_2)_n}\left\{\frac{h_2h_4\beta\delta S}{(1+S)^2}\right\}^n\,{}_2F_1\left[\begin{matrix}\mu_2+\sigma_2+n, n-\eta; & \dfrac{h_2\beta}{1+S}\\ c_1+n; & \end{matrix}\right]$$

$${}_2F_1\left[\begin{matrix}\mu_4+\sigma_4+n, n-\eta; & \dfrac{h_4\delta}{1+S}\\ c_2+n; & \end{matrix}\right].$$

Setting $b_1=0$, $b_2=0$, and $b_4=0$ in both sides of (3.7), we find a result substantially equivalent to Meixner's result [7, eqn. (19b), p. 345]

$$(4.17)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2F_1\left[\begin{matrix}\mu_2+\sigma_2, n-\eta;\\ c_1;\end{matrix}\, h_2\beta\right]{}_2F_1\left[\begin{matrix}\mu_3+\sigma_3, \lambda+n;\\ c_2;\end{matrix}\, h_3\gamma\right]T^n =$$

$$(1-T)^{-\lambda}F_2\left[\lambda,\mu_2+\sigma_2,\mu_3+\sigma_3;c_1,c_2;\frac{h_2\beta T}{T-1},\frac{h_3\gamma}{1-T}\right].$$

5. Conclusions and Discussions. From the results obtained in the previous sections, we conclude that if the sequence of quadruple hypergeometric function $K_{12}(.)$ due to Exton [3,4] is taken in the form of the sequence

$$(5.1)\quad \left\langle K_{12}\left(a,a,a,a,b_1(n),b_2(n),b_3(n);c_1(n),c_1(n),c_2(n),c_2(n);h_1x,h_2y,h_3z,h_4t\right)\right\rangle,$$

$n=0,1,2,\ldots$

where, n is independent of $x,y,z,t,h_1,h_2.h_3$, and h_4, and $b_i(0)=b_i, \forall i=1,2,3,4$, and

$c_j(0)=c_j, \forall j=1,2.$

$H(D)$ is the compact set of quadruple hypergeometric functions

$$(5.2)\quad K_{12}\left(a,a,a,a,b_1(n),b_2(n),b_3(n)b_4(n);c_1(n),c_1(n),c_2(n),c_2(n);h_1x,h_2y.h_3z,h_4t\right),$$

when $n=0,1,2,\ldots$

D is the region given by

$$(5.3)\quad 0\le x\le\alpha, 0\le y\le\beta, 0\le z\le\gamma, 0\le t\le\delta \text{ and } 0\le\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}+\frac{t}{\delta}\le 1.$$

Then, there exists a linear degeneration map

$P^{\mu,\sigma}\{.\}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta): H(D)\to R, f$ or $\max\{|h_1\alpha|,|h_2\beta|\}<1, \max\{|h_3\gamma|,|h_4\delta|\}<1$,

defined by

$$(5.4)\quad P^{\mu,\sigma}\left\{K_{12}\left(a,a,a,a,b_1(n),b_2(n),b_3(n)b_4(n);c_1(n),c_1(n),c_2(n),c_2(n);h_1x,h_2y.h_3z,h_4t\right)\right\}$$

$$(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)=F^{a,(b_1(n),b_2(n),b_4(n)),(c_1(n),c_2(n)),\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)=$$

$$F_3\left[\mu_1+\sigma_1,\mu_2+\sigma_2,b_1(n),b_2(n);c_1(n);h_1\alpha,h_2\beta\right]F_3\left[\mu_3+\sigma_3,\mu_4+\sigma_4,b_3(n),b_4(n);c_2(n);h_3\gamma,h_4\delta\right].$$

for all $n=0,1,2,...$
When, $n=0$, (5.4) becomes equivalent to (2.1) such that

(5.5) $$p^{\mu,\sigma}\left\{K_{12}\left(a,a,a,a,b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;h_1x,h_2y.h_3z,h_4t\right)\right\}\left(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta\right)=$$

$$F^{a,(b_1,b_2,b_3,b_4),(c_1,c_2),\mu,\sigma}\left(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta\right)=$$

$$F_3\left[\mu_1+\sigma_1,\mu_2+\sigma_2,b_1,b_2;c_1;h_1\alpha,h_2\beta\right]F_3\left[\mu_3+\sigma_3,\mu_4+\sigma_4,b_3,b_4;c_2;h_3\gamma,h_4\delta\right].$$

Again, then for any complex λ and $|T|<1$, (5.4) generates various generating relations and transformation formulae by the formula

(5.6) $$\sum_{n=0}^{\infty}\frac{(\lambda)_n T^n}{n!}P^{\mu,\sigma}\left\{a,a,a,a,b_1(n),b_2(n),b_3(n)b_4(n);c_1(n),\right.$$

$$\left.K_{12}\left(c_1(n),c_2(n),c_2(n);h_1x,h_2y.h_3z,h_4t\right)\right\}\left(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta\right)$$

$$=\sum_{n=0}^{\infty}\frac{(\lambda)_n T^n}{n!}F^{a,(b_1(n),b_2(n),b_3(n),b_4(n)),(c_1(n),c_2(n)),\mu,\sigma}\left(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta\right).$$

Example 1. When $b_1(n)=b_1,b_2(n)=b_2-n,b_3(n)=b_3,b_4(n)=b_4-n;c_1(n)=c_1,$ and $c_2(n)=c_2$, then making an appeal to (3.1) and (5.6), we find the generating relation

(5.7) $$\sum_{n=0}^{\infty}\frac{(\lambda)_n T^n}{n!}P^{\mu,\sigma}\left\{K_{12}\left(a,a,a,a,b_1,b_2-n,b_3,b_4-n;c_1,c_1.c_2,c_2;h_1x,h_2y.h_3z,h_4t\right)\right\}$$

$$\left(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta\right)=(1-T)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu_2+\sigma_2)_n(\mu_4+\sigma_4)_n}{n!(c_1)_n(c_2)_n}\left\{\frac{h_2h_4\beta\delta T}{(1-T)^2}\right\}^n$$

$$F_S\left[\mu_1+\sigma_1,\mu_2+\sigma_2+n,\mu_2+\sigma_2+n,b_1,b_2,\lambda+n;c_1+n,c_1+n,c_1+n;h_1\alpha,h_2\beta,\frac{h_2\beta T}{T-1}\right]$$

$$F_S\left[\mu_3+\sigma_3+\mu_4+\sigma_4+n,\mu_4+\sigma_4+n,b_3,b_4,\lambda+n;c_2+n,c_2+n,c_2+n;h_3\gamma,h_4\delta,\frac{h_4\delta T}{T-1}\right],$$

provided that $|h_1\alpha|<r_1,|h_2\beta|<r_2,|T|<1,\frac{\beta|h_2|}{1+|T|^{-1}}<\left|\left(\frac{\beta h_2T}{T-1}\right)\right|<r_2$ such that $r_1+r_2=r_1r_2$ and $|h_3\gamma|<r_3,|h_4\delta|<r_4,\frac{\delta|h_4|}{1+|T|^{-1}}<\left|\left(\frac{\delta h_4T}{T-1}\right)\right|<r_4$ such that $r_3+r_4=r_3r_4$.

Example 2. When $b_1(n)=\lambda+n,b_2(n)=b_2,b_3(n)=\lambda+n,b_4(n)=b_4;c_1(n)=c_1,$ and $c_2(n)=c_2$, then making an appeal to (3.11) and (5.6), we find the transformation formula

$$(5.8)\ \sum_{n=0}^{\infty}\frac{(\lambda)_n T^n}{n!}P^{\mu,\sigma}\left\{K_{12}\left(a,a,a,a,\lambda+n,b_2,\lambda+n,b_4;c_1,c_1,c_2,c_2;h_1x,h_2y.h_3z,h_4T\right)\right\}$$

$$(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta)=(1-T)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu_2+\sigma_2)_n(\mu_4+\sigma_4)_n}{n!(c_1)_n(c_2)_n}\left\{\frac{h_2h_4\beta\delta T}{(1-T)^2}\right\}P^{(\mu_1,\mu_2+n,\mu_3,\mu_4+n),\sigma}$$

$$\left\{K_{12}\left(a,a,a,a,b_1,\lambda+n,b_3,\lambda+n;c_1+n,c_1+n,c_2+n,c_2+n;h_1x,\frac{h_2y}{1-T},h_3z,\frac{h_4t}{1-T}\right)\right\}$$

$$\left(h_1\alpha,\frac{h_2\beta}{1-T},h_3\gamma,\frac{h_4\delta}{1-T}\right),$$

provided that $\max\{|h_1\alpha|,|h_2\beta|\}<1, |T|<1, \max\{|h_3\gamma|,|h_4\delta|\}<1.$

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MARICHEV-SAIGO-MAEDA OPERATIONAL RELATIONSHIPS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

By

D.K. Jain

Department of Applied Mathematics, Madhav Institute of Techonology & Science, Gwalior-474 005, Madhya Pradesh, India

E-mail:jain_dkj@yahoo.co.in

Renu Jain

School of Mathematics and Allied Sciences, Jiwaji University, Gwalior-474 011, Madhya Pradesh, India

E-mail:renujain3@rediffmail.com

and

Anjali Thakur

School of Mathematics and Allied Sciences, Jiwaji University, Gwalior-474 011, Madhya Pradesh, India

E-mail:vitm.tanjali@gmail.com

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ABSTRACT

In this paper, we introduce product of generalized hypergeometric functions like I-function, Mittag-Leffler function and Bessel function by using Marichev-Saigo-Maeda fractional integral operator and we find out some image formulae by employing generalized hypergeometric function and generalized fractional integral operator due to Marichev-Saigo-Maeda involving Appell's function $F_3(.)$.

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Key words and Phrases: Marichev-Saigo-Maeda fractional integral operator, Fox's H–function, I–function, Mittag-Liffler function, Generalized hypergeometric functions, Appell's function F_3.

1. Introduction. Let $\mu,\mu',\upsilon,\upsilon',\eta\in\mathbb{C}(\Re(\eta)>0)$ and $x>0$, then the generalized fractional integral operator is defined as :

(1.1) $$\left(I_{0,x}^{\mu,\mu',\upsilon,\upsilon',\eta}f\right)(x)=\frac{x^{-\mu}}{\Gamma\eta}\int_0^x(x-t)^{\eta-1}t^{-\mu'}F_3\left(\mu,\mu',\upsilon,\upsilon';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)dt,$$

(1.2) $$\left(I_{x,\infty}^{\mu,\mu',\upsilon,\upsilon',\eta}f\right)(x)=\frac{x^{-\mu'}}{\Gamma\eta}\int_x^\infty(t-x)^{\eta-1}t^{-\mu}F_3\left(\mu,\mu',\upsilon,\upsilon';\eta;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)dt,$$

where $F_3(.)$ is Appell's function (also see Erdélyi [2]) defined by

(1.3) $F_3(\mu,\mu',\upsilon,\upsilon';\eta;x,y)=\sum_{m,n=0}^{\infty}\frac{(\mu)_m(\mu')_n(\upsilon)_m(\upsilon')_n}{(\eta)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!};\left(\max\{|x|,|y|\}<1\right).$

The generalized fractional integral operator defined through equations (1.1) and (1.2) has been introduced by Marichev [8] and later extended and studied by Saigo and Maeda [11]. These operators are known as the Marichev-Saigo-Maeda operators and properties of these operators were studied by Saigo and Maeda [11]. These fractional integral operators have many interesting applications in various sub-fields of mathematical analysis. Kim et.al. [6] have demonstrated applications related to certain class of complex analytic functions. For instance ([7],[8],[9] and [14]) can also be referred for some basic results on fractional calculus.

In this paper, following results due to Saigo and Maeda [8] will be frequently used:

(1.4) $\left(I_{0,x}^{\mu,\mu',\upsilon,\upsilon',\eta}t^{\rho-1}\right)(x)=\frac{\Gamma(\rho)\Gamma(\rho+\eta-\mu-\mu'-\upsilon)\Gamma(\rho+\upsilon'-\mu')}{\Gamma(\rho+\upsilon')\Gamma(\rho+\eta-\mu-\mu')\Gamma(\rho+\eta-\mu'-\upsilon)}x^{\rho+\eta-\mu-\mu'-1},$

where $\Re(\eta)>0;\Re(\rho)>\max\{0,\Re(\mu+\mu'+\upsilon-\eta),\Re(\mu'-\upsilon')\}$ and

(1.5) $\left(I_{x,\infty}^{\mu,\mu',\upsilon,\upsilon',\eta}t^{\rho-1}\right)(x)=\frac{\Gamma(1-\rho-\upsilon)\Gamma(1-\rho-\eta+\mu+\mu')\Gamma(1-\rho+\mu+\upsilon'-\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\mu+\mu'+\upsilon'-\eta)\Gamma(1-\rho+\mu-\upsilon)}x^{\rho+\eta-\mu-\mu'-1}$

where $\Re(\eta)>0;0<\Re(\rho)<\max\{\Re(-\upsilon),\Re(\mu+\mu'-\eta),\Re(\mu+\upsilon'-\eta)\}$.

2. *I*-Function. The *I*-function is the generalization of *H*-function of Mathai-Saxena [7] and which was introduced by Saxena [12], while solving a dual integral equation involving sum of *H*-functions as kernel. This *I*-function is defined as:

(2.1) $I(x)=I_{p_i,q_i;r}^{m,n}[x]=I_{p_i,q_i;r}^{m,n}\left[x\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L X(s)x^{-s}ds,$

where

(2.2) $X(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right\}},$

with all other conditions already detailed in [12].

The series representation of *I*-function is as follows [12]:

$$\text{(2.3)} \quad I(x) = I^{m,n}_{p_i,q_i;r}[z] = I^{m,n}_{p_i,q_i;r}\left[z \left| \begin{matrix} (e_j,E_j)_{1,n};(e_{ji},E_{ji})_{n+1,p_i} \\ (f_j,F_j)_{1,m};(f_{ji},F_{ji})_{m+1,q_i} \end{matrix} \right. \right] = \sum_{h=1}^{m}\sum_{\upsilon=0}^{\infty} \frac{(-1)^{\upsilon} X(\xi)}{\upsilon! E_h}\left(\frac{1}{z}\right)^{\xi}$$

where $\xi = (e_h - 1 - \upsilon)/E_h,\ h = 1,2,\ldots,m$ and

$$X(\xi) = \frac{\prod_{j=1}^{m}\Gamma(f_j + F_j\xi)\prod_{j=1}^{n}\Gamma(1 - e_j - E_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1 - f_{ji} - F_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(e_{ji} + E_{ji}\xi)\right\}}.$$

2. Marichev-Saigo-Maeda Operatoional Results Involving the Product of two *I*-Functions.

Theorem 1. Let $x > 0, \Re(\rho) \geq 0$, and the parameters $\mu,\mu',\upsilon,\upsilon',\eta,\rho \in \mathbb{C}, b, c \in \Re$ such that $\Re(\beta) > 0, \Re(\delta) > 0, \Re(\eta) > 0, \Re(\rho) > \max\{0, \Re(\mu + \mu' + \upsilon - \eta), \Re(\mu' - \upsilon')\}$, $\Re(1 + \eta/(1-\alpha)) > 0$ and $i = 1,2,\ldots r$. Then

$$\text{(3.1)}\ I^{\mu,\mu',\upsilon,\upsilon',\eta}_{0,x}\left(t^{\rho-1} I^{M,N}_{P_i,Q_i;R}\left[ct^{\delta} \left| \begin{matrix} (e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i} \\ (f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i} \end{matrix} \right. \right] I^{m,n}_{p_i,q_i;r}\left[bt^{\beta} \left| \begin{matrix} (a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i} \\ (b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i} \end{matrix} \right. \right] \right)$$

$$= x^{\eta-\mu-\mu'+\rho-1}\sum_{k,l=0}^{\infty}\frac{(-1)^{l}(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{k!\,l!} I^{M,N}_{P_i,Q_i;R}\left[cx^{\delta} \left| \begin{matrix} (e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i} \\ (f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i} \end{matrix} \right. \right]$$

$$I^{m,n}_{p_i,q_i;r}\left[bx^{\beta} \left| \begin{matrix} (1-\rho+l+\mu'+\delta\xi,\beta),(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i} \\ (b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i},(1-\rho-\eta-k+\mu'+\delta\xi,\beta) \end{matrix} \right. \right].$$

Proof. Making an appeal to (1.1), (1.3), (2.1) and (2.3), we get

$$I^{\mu,\mu',\upsilon,\upsilon',\eta}_{0,x}\left(t^{\rho-1} I^{M,N}_{P_i,Q_i;R}\left[ct^{\delta} \left| \begin{matrix} (e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i} \\ (f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i} \end{matrix} \right. \right] I^{m,n}_{p_i,q_i;r}\left[bt^{\delta} \left| \begin{matrix} (a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i} \\ (b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i} \end{matrix} \right. \right] \right)$$

$$= \frac{x^{-\mu}}{\Gamma\eta}\int_0^x t^{\rho-1}(x-t)^{\eta-1}t^{-\mu'}\left(\sum_{k,l=0}^{\infty}\frac{(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{(\eta)_{k+l}}\frac{(1-t/x)^k}{k!}\frac{(1-x/t)^l}{l!} \right)$$

$$\left(\sum_{h=1}^{N}\sum_{\upsilon=0}^{\infty}\frac{(-1)^{\upsilon}X(\xi)}{\upsilon!E_h}\left(\frac{1}{ct^{\delta}}\right)^{\xi}\right)\left(\frac{1}{2\pi i}\int_L \Phi(s)\left(bx^{\beta}\right)^{-s}ds\right)dt.$$

Now changing the order of integration and summation, under the given conditions, we derive

$$I_{0,x}^{\mu,\mu',\upsilon,\upsilon',\eta}\left(t^{\rho-1}I_{P_i,Q_i;R}^{M,N}\left[ct^{\delta}\left|\begin{matrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i}\end{matrix}\right.\right]I_{p_i,q_i;r}^{m,n}\left[bt^{\beta}\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]\right)=$$

$$=\frac{x^{-\mu}}{\Gamma(\eta)}\left(\sum_{k,l=0}^{\infty}\frac{(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{(\eta)_{k+1}k!l!}\sum_{h=1}^{N}\sum_{\upsilon=0}^{\infty}\frac{(-1)^{\upsilon}X(\xi)}{\upsilon!E_hc^{\xi}}\right)\frac{1}{2\pi i}\int_L\Phi(s)b^{-s}$$

$$\left(\int_0^x t^{\rho-\mu'-\delta\xi-\beta s-1}(x-t)^{\eta-1}(1-t/x)^k(1-x/t)^l\,dt\right)ds$$

$$=\frac{x^{\eta-\mu-\mu'+\rho-1}}{\Gamma(\eta)}\sum_{k,l=0}^{\infty}\frac{(-1)^l(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l\Gamma(\eta+k+1)}{(\eta)_{k+1}k!l!}\sum_{h=1}^{N}\sum_{\upsilon=0}^{\infty}\frac{(-1)^{\upsilon}X(\xi)}{\upsilon!E_h}\left(\frac{1}{cx^{\delta}}\right)^{\xi}$$

$$\frac{1}{2\pi i}\int_L\Phi(s)\left(bx^{\beta}\right)^{-s}\frac{\Gamma(\rho-1-\mu'-\delta\xi-\beta s)}{\Gamma(\rho+\eta+k-\mu'-\delta\xi-\beta s)}ds.$$

Hence we get the required result (3.1).

Special Cases.

(i) Putting $l=0,k=0,\mu=0,\mu'=0,\upsilon=0,\upsilon'=0,\eta=1+\eta/(1-\alpha)$ and replacing x by $x/(a(1-\alpha))$ in the equation (3.1), we get the following result for I-function:

$$(3.2)\quad P_{0+}^{(\eta,\alpha)}\left(t^{\rho-1}I_{P_i,Q_i;R}^{M,N}\left[ct^{\delta}\left|\begin{matrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i}\end{matrix}\right.\right]I_{p_i,q_i;r}^{m,n}\left[bt^{\beta}\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]\right).$$

$$=\frac{x^{\eta+\rho}}{[a(1-a)]\rho}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)I_{P_i,Q_i;R}^{M,N}\left[\frac{cx^{\delta}}{[a(1-\alpha)]^{\delta}}\left|\begin{matrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i}\end{matrix}\right.\right]$$

$$I^{m,n+1}_{p_i+1,q_i+1;r}\left[\frac{bx^{\beta}}{\left[a(1-\alpha)\right]^{\beta}}\left|\begin{matrix}(1-\rho+\delta\xi,\beta)(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i},\left(-\rho+\delta\xi-\frac{\eta}{1-\alpha},\beta\right)\end{matrix}\right.\right],$$

which appears to be new.

(ii) For $p_i = p, q_i = q, P_i = P, Q_i = Q$ $(i=1,2,3,...)$ and $r=1$ the result (3.2) reduces to well known result due to Srivastava, Lin and Wang [13, p.163, eqn (12)] for H-function:

$$(3.3)\quad P^{(\eta,\alpha)}_{0+}\left(t^{p-1}H^{M,N}_{P,Q}\left[ct^{\delta}\left|\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right.\right]H^{m,n}_{p,q}\left[bt^{\beta}\left|\begin{matrix}(a_p,A_p)\\(b_Q,B_Q)\end{matrix}\right.\right]\right)$$

$$=\frac{x^{\eta+\rho}}{\left[a(1-\alpha)\right]^{\rho}}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)H^{M,N}_{P,Q;R}\left[\frac{cs^{\delta}}{\left[a(1-\alpha)\right]^{\delta}}\left|\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right.\right]$$

$$H^{m,n+1}_{p+1,q+1}\left[\frac{bx^{\beta}}{\left[a(1-\alpha)\right]^{\beta}}\left|\begin{matrix}(1-\rho+\delta\xi,\beta),(a_1,A_1),...,(a_p,A_p)\\(b_1,B_1),...,(b_Q,B_Q),\left(-\rho+\delta\xi-\frac{\eta}{1-\alpha},\beta\right)\end{matrix}\right.\right].$$

4. Mittag-Leffler Function. According to Erdélyi et al. [3] (also see Prabhaker [10]) the Swedish Mathematician Gosta Mittag-Leffler introduced the function $E_{\beta}(z)$ defined by

$$(4.1)\quad E_{\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+1)},$$

where z is a complex variable, $\Gamma(.)$ is a gamma function and $\beta \geq 0$. The Mittag-Leffler function is the direct generalization of the exponential function to which it reduces for $\beta = 1$. For $0<\beta<1$, it is interpolated between the pure exponential and a hypergeometric function $1/(1-z)$. Mittag- Leffler function naturally occurs as integral equations.

Wiman [14] studied the generalization of $E_{\beta}(z)$, that is given by

$$(4.2)\quad E_{\beta,\rho}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+\rho)};\rho\in C,R(\rho)>0,R(\beta)>0,$$

which is known as Wiman's function.

Prabhakar [10] investigated the function $E^{\gamma}_{\beta,\rho}(z)$ as

(4.3) $$E^{\gamma}_{\beta,\rho}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\beta k+\rho)}\frac{z^k}{k!};\beta,\rho,\gamma\in C,R(\rho)>0,R(\beta)>0,R(\gamma)>0.$$

The Mittag-Leffler type functions belong to family of I-functions, since they can be represented in terms of H-function:

(4.4) $$E_{\beta}(z)=H^{1,1}_{1,2}\left[-z\left|\begin{matrix}(0,1)\\(0,1),(0,\beta)\end{matrix}\right.\right];\beta\in C,R(\beta)>0,$$

(4.5) $$E_{\beta,\rho}(z)=H^{1,1}_{1,2}\left[-z\left|\begin{matrix}(0,1)\\(0,1),(1-\rho,\beta)\end{matrix}\right.\right];\beta,\rho\in C,R(\beta)>0,$$

(4.6) $$E^{\gamma}_{\beta,\rho}(z)=H^{1,1}_{1,2}\left[-z\left|\begin{matrix}(1-\gamma,1)\\(0,1),(1-\rho,\beta)\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i\Gamma\gamma}\int_{e-i\infty}^{e+i\infty}\frac{\Gamma s\Gamma(y-s)}{\Gamma(\beta-as)}(-z)^{-s};\beta,\rho,\gamma\in C,R(\beta)>0.$$

The relation connecting the Wright's function ${}_p\psi_q(z)$ ([15],[16]) and the H-function was given for the first time in the monograph of Mathai and Saxena [7, p.11, eqn. 1.7.81] as:

(4.7) $${}_p\psi_q\left[\begin{matrix}(a_1,A_1)\ldots(a_p,A_p)\\(b_1,B_1)\ldots(b_p,B_p)\end{matrix}\middle|z\right]=H^{1,p}_{p,q+1}\left[-z\left|\begin{matrix}(1-a_1,A_1),\ldots,(1-a_p,A_p)\\(0,1),(1-b_1,B_1),\ldots,(1-b_Q,B_Q)\end{matrix}\right.\right].$$

5. Marichev-Saigo-Maeda Operational Results Involving the Product of *I*-function and Mittag-Leffler Function.

Theorem 2. Let $x>0,\Re(\rho)\geq 0$, and the parameters $\mu,\mu',\upsilon,\upsilon',\eta,\rho,\gamma\in\mathbb{C}$ and $b,c\in\Re$ be such that $\Re(\gamma)>0,\Re(\eta)>0,\Re(1+\eta/(1-\alpha))>\max[0,-R(\rho)]$, $\Re(\rho)>\max\{0,$ $\Re(\mu+\mu'+\upsilon-\eta),\Re(\mu'-\upsilon')\},\beta>0,\delta>0$ and $i=1,2,\ldots r$. Then

(5.1) $$I^{\mu,\mu',\upsilon,\upsilon',\eta}_{0,x}\left(t^{p-1}I^{M,N}_{P_i,Q_i;R}\left[ct^{\delta}\left|\begin{matrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i}\end{matrix}\right.\right]E^{\gamma}_{\beta,\delta}(bt^{\beta})\right)$$

$$=\frac{1}{\Gamma(\gamma)}x^{\eta-\mu-\mu'+\rho-1}\left(\sum_{k,l=0}^{\infty}\frac{(-1)^{l}(\mu)_{k}(\mu')_{l}(\upsilon)_{k}(\upsilon')_{l}}{k!l!}\right)I_{P_i,Q_i;R}^{M,N}\left[cx^{\delta}\left|\begin{matrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i}\end{matrix}\right.\right]$$

$${}_2\Psi_2\left[\left.\begin{matrix}(\rho-1-\mu'-\delta\xi,\beta),(\gamma,1)\\(\rho,\beta),(\rho+\eta+k-\mu'-\delta\xi,\beta)\end{matrix}\right|bx^{\beta}\right].$$

Proof. Making an appeal to (1.1), (1.3), (2.3) and (4.6), we get

$$I_{0,x}^{\mu,\mu',\upsilon,\upsilon',\eta}\left(t^{\rho-1}I_{P_i,Q_i;R}^{M,N}\left[ct^{\delta}\left|\begin{matrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_1}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_1}\end{matrix}\right.\right]E_{\beta,\delta}^{\gamma}\left(bt^{\beta}\right)\right)$$

$$=\frac{x^{-\mu}}{\Gamma(\eta)}\int_0^x t^{\rho-1}(x-1)^{\eta-1}t^{-\mu'}\left(\sum_{k,l=0}^{\infty}\frac{(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{(\eta)_{k+l}}\frac{(1-t/x)^k}{k!}\frac{(1-x/t)^l}{l!}\right)$$

$$\left(\sum_{h=1}^{N}\sum_{\upsilon=0}^{\infty}\frac{(-1)^{\upsilon}X(\xi)}{\upsilon!E_h}\left(\frac{1}{ct^{\delta}}\right)^{\xi}\right)H_{1,2}^{1,1}\left[-bt^{\beta}\left|\begin{matrix}(1-\gamma,1)\\(0,1),(1-\rho,\beta)\end{matrix}\right.\right]dt.$$

Now changing the order of integration and summation, under the given conditions, we derive

$$I_{0,x}^{\mu,\mu',\upsilon,\upsilon',\eta}\left(t^{\rho-1}I_{P_i,Q_i;R}^{M,N}\left[ct^{\delta}\left|\begin{matrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i}\end{matrix}\right.\right]E_{\beta,\delta}^{\gamma}\left(bt^{\beta}\right)\right)$$

$$=\frac{x^{-\mu}}{\Gamma(\eta)}\left(\sum_{k,l=0}^{\infty}\frac{(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{(\eta)_{k+l}k!l!}\sum_{h=1}^{N}\sum_{\upsilon=0}^{\infty}\frac{(-1)^{\upsilon}X(\xi)}{\upsilon!E_hc^{\xi}}\right)\frac{1}{2\pi i\Gamma\gamma}\int_L\frac{\Gamma s\Gamma(\gamma-s)}{\Gamma(\beta-\alpha s)}b^{-s}$$

$$\left(\int_0^x t^{\rho-\mu'-\delta\xi-\beta s-1}(x-t)^{\eta-1}(1-t/x)^k(1-x/t)^1\,dt\right)ds,$$

$$=\frac{x^{\eta-\mu-\mu'+\rho-1}}{\Gamma(\eta)}\sum_{k,l=0}^{\infty}\frac{(-1)^{l}(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l\Gamma(\eta+k+1)}{(\eta)_{k+1}k!l!}\sum_{h=1}^{N}\sum_{\upsilon=0}^{\infty}\frac{(-1)^{\upsilon}X(\xi)}{\upsilon!E_h}\left(\frac{1}{cx^{\delta}}\right)^{\xi}$$

$$\frac{1}{2\pi i\Gamma\gamma}\int_L\frac{\Gamma s\Gamma(\gamma-s)}{\Gamma(\beta-\alpha s)}\left(bx^{\beta}\right)^{-s}\frac{\Gamma(\rho-1-\mu'-\delta\xi-\beta s)}{\Gamma(\rho+\eta+k-\mu'-\delta\xi-\beta s)}ds$$

$$=\frac{1}{\Gamma(\gamma)}x^{\eta-\mu-\mu'+\rho-1}\sum_{k,l=0}^{\infty}\frac{(-1)^{l}(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{k!l!}I_{P_i,Q_i;R}^{M,N}\left[cx^{\delta}\left|\begin{matrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i}\end{matrix}\right.\right]$$

$$H_{2,3}^{1,2}\left[-bt^{\beta}\left|\begin{matrix}(1-\rho+1+\mu'+\delta\xi,\beta),(1-\gamma,1)\\(0,1),(1-\rho,\beta),(1-\rho-\eta-k+\mu'+\delta\xi,\beta)\end{matrix}\right.\right].$$

Hence we get the required result (5.1).

Special Cases.

(i) Putting $l=0,k=0,\mu=0,\mu'=0,\upsilon=0,\upsilon'=0,\eta=1+n(1-\alpha)$ and replacing x by $x/(a(1-\alpha))$ in relation (5.1), we get the following result for I function:

$$(5.2)\ P_{0+}^{(\eta,\alpha)}\left(t^{\rho-1}I_{P_i,Q_i;R}^{M,N}\left[ct^{\delta}\left|\begin{matrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i}\end{matrix}\right.\right]E_{\beta,\delta}^{\gamma}(bt^{\beta})\right)=\frac{1}{\Gamma\gamma}\frac{x^{\eta+\rho}}{[a(1-\alpha)]^{\rho}}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)$$

$$I_{P_i,Q_i;R}^{M,N}\left[\frac{cx^{\delta}}{[a(1-\alpha)]^{\delta}}\left|\begin{matrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i}\end{matrix}\right.\right]{}_2\Psi_2\left[\begin{matrix}(\rho-\delta\xi,\beta),(\gamma,1)\\(\rho,\beta),\left(1+\rho+\frac{\eta}{1-\alpha}-\delta\xi,\beta\right)\end{matrix}\left|\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right.\right],$$

(ii) Now for $p_i=p,q_i=q,P_i=P,Q_i=Q$ $(i=1,2,3,...)$ and $r=1$ the result (5.2) reduces to well known result due to Srivastava, Lin and Wang [13, p.165, eqn 21] for H-function:

$$(5.3)\ P_{0+}^{(\eta,\alpha)}\left(t^{\rho-1}H_{P,Q}^{M,N}\left[ct^{\delta}\left|\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right.\right]E_{\beta,\delta}^{\gamma}(bt^{\beta})\right)=$$

$$=\frac{1}{\Gamma(\gamma)}\frac{x^{\eta+\rho}}{[a(1-\alpha)]^{\rho}}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)\left(H_{P,Q:R}^{M,N}\left[\frac{cx^{\delta}}{[a(1-\alpha)]^{\delta}}\left|\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right.\right]\right.$$

$$\left.{}_2\Psi_2\left[\begin{matrix}(\rho-\delta\xi,\beta),(\gamma,1)\\(\rho,\beta),\left(1+\rho+\frac{\eta}{1-\alpha}-\delta\xi,\beta\right)\end{matrix}\left|\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right.\right]\right).$$

6. Bessel Function. The Bessel function of the first kind $J_v(x)$ is defined for complex $x\in C,x\neq 0$ and $v\in C,R(v)>-1$ by

$$(6.1)\quad J_v(x)=\left(\frac{x}{2}\right)^{v}\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^k}{k!\Gamma(\upsilon+k+1)} \qquad (22)$$

and its H-function representation is given by

$$(6.2)\qquad J_{\upsilon}(x)=\left(\frac{x}{2}\right)^{\upsilon} H_{0,2}^{1,0}\left[\frac{x^2}{4}\middle|\begin{matrix}-\\(0,1),(-\upsilon,1)\end{matrix}\right],\upsilon\in C, R(\upsilon)>0.$$

For the asymptotic behavior and other properties in detail see for instance ([1],[5] and [6]).

7. Relations of the Product of *I*-Function and Bessel Functions using Marichev-Saigo-Maeda Operational Results Involving the Product of *I*-Function and Bessel Functions.

Theorem 3. Let $x>0$ and the parameters $\mu,\mu',\upsilon,\upsilon',\eta,\gamma,\upsilon\in\mathbb{C}$ and $c\in\Re$ be such that $\Re(\eta)>0,\Re(\gamma+\upsilon)>0,\Re(1+\eta/(1-\alpha))>0,\beta>0,\delta>0$ and $i=1,2,\ldots,r$. Then

$$(7.1)\qquad I_{0,x}^{\mu,\mu',\upsilon,\upsilon',\eta}\left(\left(\frac{t}{2}\right)^{\gamma-1} I_{P_i,Q_i;R}^{M,N}\left[c\left(\frac{t}{2}\right)^{\delta}\middle|\begin{matrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i}\end{matrix}\right]J_{\alpha}(t)\right)$$

$$=\frac{x^{\gamma+\alpha+\eta-\mu'-\mu-1}}{2^{\gamma+\alpha-1}}\left(\sum_{k,l=0}^{\infty}\frac{(-1)^l(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{(\eta)_{k+1}\,k!\,l!}\right)I_{P_i,Q_i;R}^{M,N}\left[c\left(\frac{x}{2}\right)^{\delta}\middle|\begin{matrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i}\end{matrix}\right]$$

$$ {}_1\Psi_2\left[\begin{matrix}(\gamma+\alpha-1-\mu'-\delta\xi,2)\\(1+\alpha,1),(\gamma+\alpha+\eta+k-\mu'-\delta\xi,2)\end{matrix}\middle|\frac{-x^2}{4}\right].$$

Proof. Making an appeal to (1.1), (1.3), (2.3) and (6.2), we derive

$$I_{0,x}^{\mu,\mu',\upsilon,\upsilon',\eta}\left(\left(\frac{t}{2}\right)^{\gamma-1} I_{P_i,Q_i;R}^{M,N}\left[c\left(\frac{t}{2}\right)^{\delta}\middle|\begin{matrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i}\end{matrix}\right]J_{\alpha}(t)\right)$$

$$=\frac{x^{-\mu}}{\Gamma(\eta)}\int_0^x\left(\frac{t}{2}\right)^{\gamma-1}(x-t)^{\eta-1}t^{-\mu'}\left(\sum_{k,l=0}^{\infty}\frac{(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{(\eta)_{k+1}}\frac{(1-t/x)^k}{k!}\frac{(1-x/t)^l}{l!}\right)$$

$$\left(\sum_{h=1}^{N}\sum_{\upsilon=0}^{\infty}\frac{(-1)^{\upsilon}X(\xi)}{\upsilon!E_h}\left(\frac{1}{c(t/2)^{\delta}}\right)^{\xi}\right)\left(\frac{t}{2}\right)^{\alpha}H_{0,2}^{1,0}\left[\frac{t^2}{4}\middle|\begin{matrix}-\\(0,1),(-\alpha,1)\end{matrix}\right]dt,$$

$$=\frac{x^{-\mu}}{\Gamma(\eta)}\left(\sum_{k,l=0}^{\infty}\frac{(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{(\eta)_{k+1}\,k!\,l!}\sum_{h=1}^{N}\sum_{\upsilon=0}^{\infty}\frac{(-1)^{\upsilon}X(\xi)}{\upsilon!E_h c^{\xi}}\right)$$

$$\frac{1}{2\pi i}\int_L \frac{\Gamma s}{\Gamma(1+\alpha-s)}\left(\int_0^x (t/2)^{\gamma+\alpha-1}(x-t)^{\eta-1}t^{-\mu'}\left(t^2/4\right)^{-s}(t/2)^{-\delta\xi}(1-t/x)^k(1-x/t)^l\,dt\right)ds,$$

$$=\frac{x^{\gamma+\alpha+\eta-\mu'-\mu-1}}{2^{\gamma+\alpha-1}}\left(\sum_{k,l=0}^{\infty}\frac{(-1)^l(\mu)_k(\mu')_l(\upsilon)_k(\upsilon')_l}{(\eta)_{k+1}\,k!\,l!}\right)I^{M,N}_{P_i,Q_i;R}\left[c\left(\frac{x}{2}\right)^{\delta}\middle|\begin{matrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i}\end{matrix}\right]$$

$$H^{1,1}_{1,3}\left[\frac{x^2}{4}\middle|\begin{matrix}(1-\gamma-\alpha+1+\mu'+\delta\xi,2)\\(0,1),(-\alpha,1),(1-\gamma-\alpha-\eta-k+\mu'+\delta\xi,2)\end{matrix}\right].$$

Thus we establish the required result (7.1).

Special Cases.

(i) Putting $l=0, k=0, \mu=0, \mu'=0, \upsilon=0, \upsilon'=0, \eta=1+\eta/(1-\alpha)$ and replacing x by $x/(a(1-\alpha))$ in the equation (7.1), we get the following result for I-function:

$$(7.2)\quad P_{0+}^{(\eta,\alpha)}\left(\left(\frac{t}{2}\right)^{\gamma-1}I^{M,N}_{P_i,Q_i;R}\left[c\left(\frac{t}{2}\right)^{\delta}\middle|\begin{matrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i}\end{matrix}\right]J_{\alpha}(t)\right)$$

$$=\frac{\Gamma(1+\eta/(1-\alpha))x^{\gamma+\alpha+\eta}}{2^{\gamma=\alpha-1}\left[a(1-\alpha)\right]^{\gamma+\alpha}}I^{M,N}_{P_i,Q_i;R}\left[\frac{cs^{\delta}}{\left[2a(1-\alpha)\right]^{\delta}}\middle|\begin{matrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i}\end{matrix}\right]$$

$${}_1\Psi_2\left[\begin{matrix}(\gamma+\alpha-\delta\xi,2)\\(1+\alpha,1),(1+\gamma+\alpha+\eta/(1-\alpha)-\delta\xi,2)\end{matrix}\middle|\frac{-x^2}{4a^2(1-\alpha)^2}\right],$$

which appears to the new

(ii) For $p_i=p, q_i=q, P_i=P, Q_i=Q$ $(i=1,2,3,...)$ and $r=1$ the result (7.2) reduces to well known result due to Srivastava, Lin and Wang [13 p.166 eqn (24)] for H-function:

$$(7.3)\quad P_{0+}^{(\eta,\alpha)}\left(t^{\rho-1}H^{M,N}_{P,Q}\left[ct^{\delta}\middle|\begin{matrix}(e_p,E_P)\\(f_Q,F_Q)\end{matrix}\right]E^{\gamma}_{\beta,\delta}\left(bt^{\beta}\right)\right)$$

$$=\frac{\Gamma(1+\eta/(1-\alpha))x^{\gamma+\alpha+\eta}}{2^{\gamma+\alpha-1}\left[a(1-\alpha)\right]^{\gamma+\alpha}}H^{M,N}_{P_i,Q_i;R}\left[\frac{cs^{\delta}}{\left[2a(1-\alpha)\right]^{\delta}}\middle|\begin{matrix}(e_p,E_P)\\(f_Q,F_Q)\end{matrix}\right]$$

$$ {}_1\psi_2\left[\begin{matrix}(\gamma+\alpha-\delta\xi,2)\\(1+\alpha,1),(1+\gamma+\alpha+\eta/(1-\alpha)-\delta\xi,2)\end{matrix}\middle|\frac{-x^2}{4a^2(1-\alpha)^2}\right]. $$

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Dr. R.C. Singh Chandel
Publisher/Editor
rc_chandel@yahoo.com